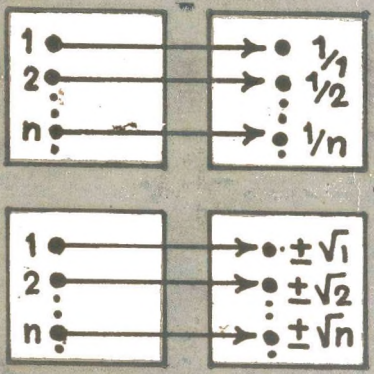
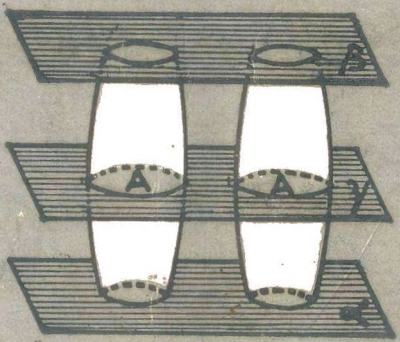
MATHEMATICS AND STATISTICS

FOR

Std. XI (HIGHER SECONDARY)

[FIRST YEAR (JUNIOR COLLEGE)]





D. S. AGASHE

T. G. KULKARNI

PAPER I

GEOMETRY

TRIGONOMETRY

AND STATISTICS

A TEXT BOOK SANCTIONED BY MAHARASHTRA STATE BOARD OF SECONDARY EDUCATION

MATHEMATICS & STATISTICS

For

Standard XI – Higher Secondary Schools

[First Year – Junior Colleges]

Paper I

Geometry, Trigonometry and Statistics

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BY

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PREFACE TO THE SECOND EDITION

While bringing out this second edition of the book we have removed the printing mistakes and carried out the suggestions made to us by the authorities of the Maharashtra State Board of Secondary Education, Pune. Blocks of some figures were newly prepared to make the figures more appropriate and instructive. The matter in the Chapters on Vectors and on mensuration of solids has been rearranged to maintain the continuity of the subject matter. $l(AB)$, $l(BC)$, ... etc. are denoted simply by AB, BC, ... etc. as per revised instructions of the Board. Short questions similar to those set by the Board in the Model Paper have been added at appropriate places. Summary of important results in various chapters has been added at the end to increase the utility of the book and to help the students to take a review of the whole subject at the end.

Illustrative examples and Exercises in various Chapters have been printed in smaller type (8 point) to reduce the number of pages so as to enable us to reduce the price of the book. It may be noted that the price of the book has been reduced from Rs. 13.50 to Rs. 10 voluntarily by the publishers.

We hope that this edition will be more economical and more useful to students for preparing the subject for the examination.

We sincerely thank the Manager Shri. S. N. Andhrutkar and the staff of Maharashtra Sahakari Mudranalaya for helping us in all respects in the printing of this book.

Suggestions for the improvement of the book will be gratefully received.

Bombay
June 1977

D. S. Agashe
T. G. Kulkarni

PREFACE TO THE FOURTH EDITION

Authors have taken the opportunity of correcting a few misprints while bringing out this edition.

Authors

PLEDGE

" India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone lies my happiness. "

PREFACE TO THE FIRST EDITION

The Government of Maharashtra have accepted the $10 + 2 + 3$ pattern of Education and have formulated the new courses of instruction in various subjects to be implemented progressively from June 1975. This has naturally led to the necessity of preparing new books catering to the needs of students offering the various subjects according to the new pattern.

We have written this book covering the course in Paper I-Geometry, Trigonometry and Statistics — of Mathematics and Statistics for Standard XI. It has been possible for us to write this book within a couple of months only because we could freely make use of the relevant matter on various topics from our existing text-books on Mathematics written for the Pre-Degree and F. Y. Classes of our Universities in the State of Maharashtra.

We have adhered very closely to the Syllabus laid down by the State Board of Secondary Education and have also made use of the useful guide-lines circulated to the Authors for use in the preparation of text-books. We have done our best in judiciously arranging the various topics in this Paper in a coherent manner so as to enable the students to study the various inter-related branches of the subject effectively by making use of their knowledge in one branch in the study of other branches. We have tried our best to make the subject as lucid and interesting as possible by providing a fairly large number of typical solved examples along with appropriate figures and diagrams together with a sufficient number of Exercises for practice.

We have also given, for immediate reference, the syllabus in the subject to guide the students for necessary preparation.

We earnestly hope that this book will meet the requirements of students studying for this Paper and instil in them a keen desire to persue the further study of Mathematics.

Our thanks are due to Shri M. S. Date and the staff of Yashwant Mudranalaya for helping us in all respects in the printing of this book.

Bombay,
10th December, 1974

D. S. Agashe
T. G. Kulkarni

SYLLABUS IN MATHEMATICS AND STATISTICS

For Standard XI

Paper I

Geometry, Trigonometry and Statistics

SECTION I

1. Vectors : Vector as a directed line segment. Position vector of a point. Addition and subtraction of vectors. Multiplication of a vector by a scalar. Section formula. Collinear points and concurrent lines. Simple applications to geometry.

2. Co-ordinate Geometry : Rectangular Cartesian co-ordinate system. Cartesian and polar co-ordinates of a point. Interpretation of position vector as an ordered pair. Distance between two points. Co-ordinates of a point of division. Equation to a locus satisfying a given condition. Change of origin. Derivation of equation to a line in the following standard forms :

(i) $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$,

(ii) $y - y_1 = m(x - x_1)$,

(iii) $y = mx + c$,

(iv) $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$,

(v) $\frac{x}{a} + \frac{y}{b} = 1$,

(vi) $x\cos\alpha + y\sin\alpha = p$.

The linear equation $ax + by + c = 0$.

Angle between two lines. Condition that two lines be perpendicular or parallel. Intersection of two lines. Distance of a point from a line.

3. Mensuration : Parallelopipeds and pyramids—their surface areas and volumes. Cylinders and Cones. Surface areas and volumes of right circular cylinder and right circular cone. Cavalieri's principle. Volumes of prisms, Sphere, Spherical cap and Spherical segment.

(Emphasis is on geometrical properties and mensuration)

SECTION II

4. Trigonometry : Angles and their measures—sexagesimal and radian. Circular functions—their domain, range and periodicity. Graphs of circular functions. Simple trigonometrical identities. Derivation of $\cos(A-B)$ from distance formula. Derivation of the formulae : $\cos(A+B)$, $\sin(A \pm B)$, $\sin 2A$, $\cos 2A$, $\tan 2A$, $\sin A/2$, $\cos A/2$. Standard formulae for conversion of sums into products and products into sums. Sine and cosine laws for a triangle. Application to problems on heights and distances.

5. Statistics : Population. Frequency distribution and frequency diagrams. Ungrouped and grouped data. Averages—their merits and demerits. Measures of location and dispersion. Deviation from an average value. Absolute deviation from the mean and standard deviation. Variance.

CONTENTS

Chapter		*Pages*
1.	Measurement of Angles ..	1 — 9
2.	Circular Functions and Identities ..	10 — 33
3.	Distance and Section Formulae ..	34 — 47
4.	Locus and Translation of Axes ..	48 — 58
5.	Straight Lines ..	59 — 89
6.	Formulae for Compound Angles ..	90 — 104
7.	Properties of a Triangle ..	105 — 117
8.	Vectors ..	118 — 150
9.	Prisms and Cylinders ..	151 — 174
10.	Pyramids and Cones ..	175 — 197
11.	Sphere ..	198 — 210
12.	Frequency Distributions ..	211 — 235
13.	Measures of Central Tendency ..	236 — 251
14.	Measures of Dispersion ..	252 — 269
	Summary of Important Results ..	270 — 279
	Tables of Natural Sines and Tangents ..	280 — 284

CHAPTER 1

Measurement of Angles

(1) **Angle** : You are already familiar with the definition of an angle as the union of two rays which have a common end point but which are not collinear. (See Fig. 1·1). The two rays are called the **sides** or arms of the angle and their common end point is called the **vertex** of the angle. Thus, according to this definition, the measure of an angle is always positive and it lies between 0° and 180°. To extend the concept of an angle so that angles of any magnitude and sign can be considered the measure of an angle is now defined as follows.

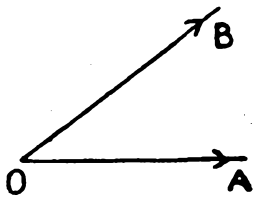


Fig. 1·1

Definition : The measure of an angle is the amount of rotation, in a plane, of a ray about its end point. This measure is positive if the rotation is in the anti-clockwise direction and is negative if the rotation is in the clock-wise direction. The measure is zero if the rotation is nil.

Thus, when a ray OA, initially coinciding with the ray OX, rotates about O in the anti-clockwise direction and takes up the final position OP, it is said to generate the $\angle AOP$ of a positive measure : $m(\widehat{OA,OP})$ or $m(\angle AOP)$. Similarly, if the ray OA, initially coinciding with OX, rotates about O in the clockwise direction and takes up the final position OP' it is said to generate the $\angle AOP'$ of negative measure : $m(\widehat{OA,OP'})$ or $m(\angle AOP')$. If the ray does not undergo any rotation then the angle is said to be of zero measure : $m(\widehat{OA,OA})$ or $m(\angle AOA)$. Thus we define a directed angle to be an ordered pair (initial ray, final ray) of coterminal rays associated with anti-clockwise or clockwise rotation from the initial ray giving the positive or negative measure, of the angle, respectively.

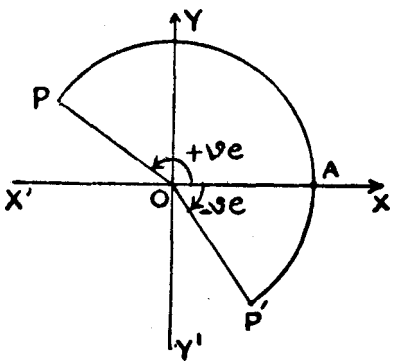


Fig. 1·2

One complete rotation of the ray about its end point generates an angle of measure $+360°$ or $-360°$ according as the rotation is in the anticlockwise or clockwise direction. Similarly, one half of a complete rotation of a ray generates a straight angle of measure $180°$ or $-180°$ according as the rotation is in the anti-clockwise or clockwise direction. Similarly, one-fourth of a complete rotation in the positive i.e. anti-clockwise direction generates an angle of measure $90°$, which is called one right angle.

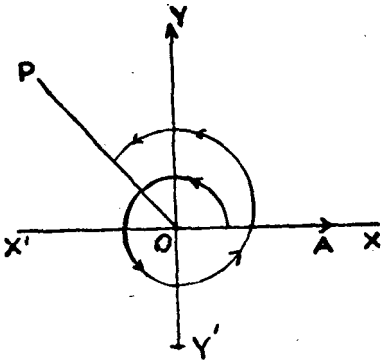


Fig. 1·3

It is obvious that n complete revolutions generate an angle of measure $4n$ right angles. Now consider figures 1·3 and 1·4.

In Fig. 1·3, the ray OP, first coinciding with OA, rotates in the anti-clockwise direction and generates an angle of measure $5\frac{1}{2}$ right angles i. e. 495°.

In Fig. 1·4, the ray OP, first coinciding, with OA, rotates in the clockwise direction and generates an angle of measure $-7\frac{1}{2}$ right angles i. e. $-675°$.

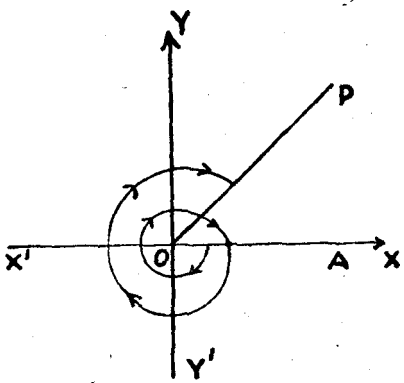


Fig. 1·4

(2) **Measurement of angles** : For a comparison of different angles. we have to adopt suitable units for giving the magnitudes of angles. We consider, here, two systems for the measurement of an angle.

(I) **Sexagesimal System** : In this system a right angle, which is generated by one-fourth of a complete revolution, is considered as a unit. For measuring smaller angles, a right angle is divided into 90 equal parts, each part being called a degree; a degree is subdivided into 60 equal parts called minutes and a minute is further divided into 60 equal parts called seconds.

One degree is written as : 1°. One minute is written as : 1′ and one second is written as 1″.

We thus have 1 rt. angle $= 90°$, $1° = 60'$, $1' = 60''$.

(II) Circular Measure : In this system the unit for measuring angles is called a radian which is defined as follows :

Definition : A radian is the measure of the angle subtended at the centre of a circle by its arc equal to the length of the radius of the circle.

One radian is written as : 1^c.

Theorem : A radian measure is independent of any circle that is used in its definition.

[In order to prove the result we assume the following two results: (i) The ratio of the length of the circumference of a circle to its diameter is constant and is denoted by π whose approximate value in vulgar fractions is taken as $\frac{22}{7}$, or 3·1416 in decimals, correct to four places of decimals. Thus the circumference of a circle of radius r is equal to $2\pi r$. (ii) The lengths of arcs of a circle are proportional to the measures of the angles subtended by them at its centre.]

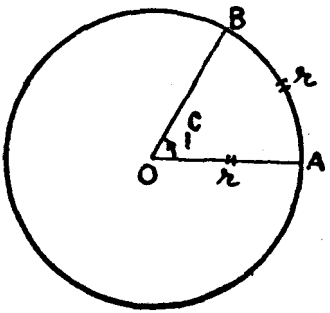


Fig. 1·5

Proof : Let O be the centre of any circle whose radius is r; let AB be an arc of the circle, whose length is equal to r.

$\because$ the measures of the angles at the centre of a circle are proportional to the lengths of the arcs which subtend them.

$$\therefore \frac{m\,(\angle AOB)}{l\,(\text{the arc } AB)} = \frac{4 \text{ right angles}}{l\,(\text{the circumference})},$$

$$\text{i. e. } \frac{m\,(\angle AOB)}{r} = \frac{4 \text{ right angles}}{2\pi r}$$

$$\therefore m\,(\angle AOB) = \frac{2}{\pi} \text{ right angles} = \text{a constant.}$$

(3) Relation between the circular measure and the sexagesimal measure : The circular measure of an angle is the number of radians contained in it. Consider a circle of unit radius. An arc of one unit length of the unit circle will subtend an angle of 1 radian at its centre. The length of the circumference of the unit circle is 2π. The length of

the minor arc AB of the unit circle (Fig. 1·6) is of length $\frac{\pi}{2}$ units and hence

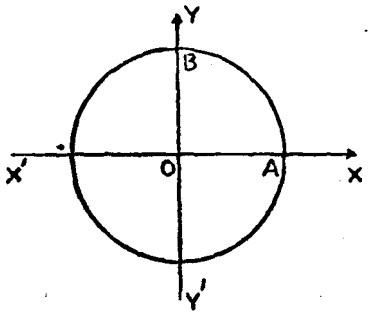


Fig. 1·6

$$m(\angle AOB) = \frac{\pi}{2} \text{ radians.} \quad \ldots (i)$$

Also $m(\angle AOB)$, being a right angle, $= 90°$. ... (ii)

From (i) and (ii), we have

$$\frac{\pi}{2} \text{ radians} = 90°.$$

$$\therefore \quad 1^c = \left(\frac{180}{\pi}\right)°. \quad \ldots (iii)$$

Taking $\pi = \frac{22}{7}$ approximately, we get

$$1^c = 180 \times \frac{7}{22} \text{ degrees, approximately}$$

$$= \frac{630}{11} \text{ degree, approximately}$$

$$= 57\frac{3}{11} \text{ degrees, approximately.}$$

If we take $\pi = 3·142$, correct to three places of decimals we can show that

$$1^c = 57° 17' 45'' \text{ approximately.}$$

From (i) and (ii) we also have

$$1° = \left(\frac{\pi}{180}\right) \text{ radians.} \quad \ldots \quad \ldots (iv)$$

Formulae (iii) and (iv) help us to transform an angle measured in radians in terms of degrees and vice versa.

Illustrative Examples :

Ex. 1 : Express in radians the measures of angles

(i) 540°, (ii) 120° and (iii) 108°.

We have $180° = \pi$ radians; hence

(i) $540° = 3\pi$ radians,

(ii) $120^\circ = \frac{120 \times \pi}{180}$ radians $= \frac{2\pi}{3}$ radians.

(iii) $108^\circ = \frac{108}{180} \times \pi$ radians $= \frac{3}{5}\pi$ radians.

Ex. 2. Express in degrees, minutes and seconds, the angles of measures

(i) $\frac{5\pi}{6}$ radians, (ii) $\frac{3\pi}{8}$ radians and (iii) $\frac{17\pi}{7}$ radians.

We have π radians $= 180^\circ$; hence

(i) $\frac{5\pi}{6}$ radians $= \frac{5}{6} \times 180^\circ = 150^\circ$.

(ii) $\frac{3\pi}{8}$ radians $= \frac{3}{8} \times 180^\circ = \frac{135^\circ}{2} = 67^\circ\ 30'$ and

(iii) $\frac{17\pi}{7}$ radians $= \frac{17}{7} \times 180^\circ = \frac{3060}{7}$ degrees

$= 437^\circ\ 8'\ 34\frac{2}{7}''$.

Note : The above examples show how an angle given in sexagesimal measure can be converted into circular measure and vice versa. It may be useful to remember the measures of the following angles in degrees and radians.

degrees :	18	30	36	45	60	90	120	135
radians :	$\pi/10$	$\pi/6$	$\pi/5$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$

(4) Representation of real numbers by points on the unit circle : Consider the unit circle with the origin O as the centre. (Fig. 1·6) The length of the circumference of this circle is 2π. Let the point A represent the number zero. Obviously the same point A represents each of the + ve numbers 2π, 4π,..., $2n\pi$ since the same point A is arrived at by going round the circle in the anti-clockwise direction once, twice..., n times starting always from the initial point A. The same point A also represents the − ve the numbers -2π, -4π,..., $-2n\pi$, since the same point A in this case is arrived at by going round the circle in the clockwise direction once, twice,..., n times respectively. In general, if x is a real number such that $|x| < 2\pi$ and is represented by the point P on the unit circle, then the length of the arc AP, measured from the initial point A is $|x|$ units. Obviously the same point P represents the numbers $x \pm 2n\pi$, since the point P is reached by going round the circle n times (starting from P) in the anticlockwise or clockwise directions, respectively, n being a + ve integer. It thus follows that the correspondence from a set of a real numbers to the points on the circumference of the unit circle is many-one. Thus the whole set of real numbers is represented by points on a unit circle.

(5) **The length of an arc of a circle :** Let ACB be an arc of a circle with centre O and radius r units. Let the length of the arc AC be equal to r units so that $m\,(\angle AOC) = 1$ radian; and let the circular measure of the angle AOB be θ^c or θ radians. (The letter 'θ' is a Greek letter pronounced as "theta"). Let the length of the arc ACB be denoted by s.

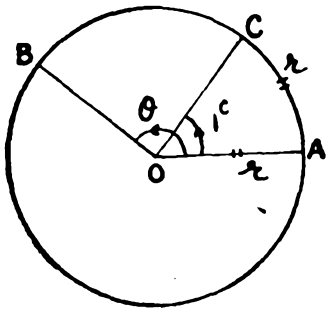


Fig. 1·7

Then by Geometry

$$\frac{l\,(\text{arc } ACB)}{m\,(\angle AOB)} = \frac{l\,(\text{arc } AC)}{m\,(\angle AOC)},$$

$$\text{i.e. } \frac{s}{\theta} = \frac{r}{1} \text{ or } s = r\theta.$$

Thus the length of an arc of a circle is radius multiplied by the number of radians contained in the angle subtended by the arc at the centre. It should be noted that the angle must be measured in radians for this purpose.

Ex. Find the lengths of the arcs of a circle subtending angles of measure (i) 120°, (ii) 225° at the centre, the radius of the circle being 12 cms.

Ans : (i) 8π cms, (ii) 15π cms.

(6) The area of a sector of a circle: A sector of a circle is the part of the circle bounded by the two radii and the arc between them; thus in figure 1·8 the shaded part AOB is a sector.

We shall assume here that (i) the area of a circle of radius r is πr^2 and (ii) the areas of sectors of the same circle are proportional to their corresponding arcs.

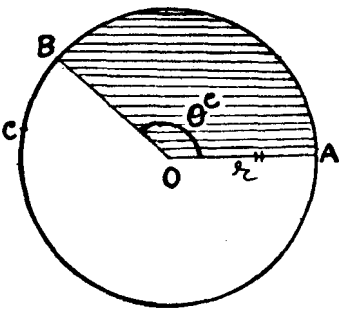


Fig. 1·8

Hence $\dfrac{\text{the area of the sector } AOB}{\text{the area of the circle}} = \dfrac{r\theta}{2\pi r}$, where the measure of the angle $AOB = \theta$ radians and r is the radius of the circle.

$\therefore$ the area of the sector $AOB = \dfrac{\theta}{2\pi} \cdot$ (the area of the circle)

$$= \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2\theta$$

where the angle θ is expressed in radians.

Illustrative Examples

Ex. 1. If the circular measure of each angle of a regular polygon is $5\pi/6$, find the number of its sides.

The sum of all the interior angles of a regular polygon of n sides is $(2n - 4)$ rt. angles.

Hence each angle of the regular polygon of n sides is $\frac{(2n-4)\pi}{2n}$,

$\therefore$ by the given data we have $\frac{(2n-4)\pi}{2n} = \frac{5\pi}{6}$, $\therefore$ $n = 12$.

Ex. 2. The measures of angles of a quadrilateral are in *A. P.* and the measure of its greatest angle is double the measure of its least angle. Find the least angle in radians.

Let the angles in degrees be $a - 3d$, $a - d$, $a + d$, $a + 3d$.

Since their sum is 360°, we have $4a = 360$, $\therefore$ $a = 90$.

Since $a + 3d = 2(a - 3d)$, we have $9d = a = 90$, $\therefore$ $d = 10$.

$\therefore$ the least angle, $a - 3d$, $= 90 - 30 = 60$ degrees i. e. $\pi/3$ radians.

Ex. 3. Find the measures of angles turned through, per minute, by (i) the second hand, (ii) the minute hand and (iii) the hour hand.

In one minute, the second hand rotates through one complete revolution i. e. through 360 degrees while in the same time the minute hand rotates through 1/60th of a complete revolution i. e. through 6 degrees only. We also observe that when the minute hand rotates through 360°, the hour hand rotates through 30 degrees only. Hence the ratio of rotation of the hour hand to that of the minute hand is 5 : 60 i. e. 1 : 12. Hence as the minute hand rotates through 6°, the hour hand will rotate through $6 \times \frac{1}{12} = \frac{1}{2}$ of a degree.

Ex. 4. The length of an arc AB of a circle is 55 cms. The measure of angle AOB is 150°, O being the centre of the circle. Find the area of the sector AOB. (Take $\pi = 22/7$).

Let the radius of the circle be r cms.

Then l (arc AB) $= r\theta$, where m (angle AOB) $= \theta$ radians.

But m angle AOB) $= 150° = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$ radians.

Hence $r \times \frac{5\pi}{6} = 55$ cms.

$\therefore$ $r = 55 \times \frac{6}{5\pi} = 55 \times \frac{6}{5} \times \frac{7}{22} = 21$ cms.

Therefore, the area of the sector AOB

$= \frac{1}{2} r^2 \theta = \frac{1}{2} \times 21 \times 21 \times \frac{5\pi}{6}$ sq. cms.

$= \frac{1}{2} \times 21 \times 21 \times \frac{5}{6} \times \frac{22}{7}$ sq. cms.

$= \frac{1155}{2}$ or $577\frac{1}{2}$ sq. cms.

EXERCISE 1

(1) Express each of the following angles in radian measure :

(i) 75°, (ii) 195°, (iii) 35° 30′, (iv) – 300°, (v) – 825°.

(2) Express each of the following angles in degree measure :

$$(\text{i})\ \frac{4\pi^c}{3},\ (\text{ii})\ \frac{21\pi^c}{4},\ (\text{iii})\ \frac{-\pi^c}{12},\ (\text{iv})\ \left(-4\frac{2}{5}\right)^c.$$

(Take $\pi = 22/7$).

(3) The measures of angles of a triangle are in the ratios 2:3:4. Find the measures of the angles (i) in degrees and (ii) in radians.

(4) If the measures of two angles of a triangle are 1·2 and 0·8 radians, find approximately the third angle in degrees. (Take $\pi = 22/7$).

(5) If the measures of two angles A and B be such that $m(A + B) = (\pi/30)^c$ and $m(A - B) = 4°$, find the circular measures of $\angle A$ and $\angle B$.

(6) The measures of angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as 45 is to π. Find the measures of the angles in degrees.

(7) Find the measure of the acute angle in radians and degrees between the hour hand and the minute hand of a clock at quarter past one.

(8) At what time between three and four are the minute and hour hands of a clock overlapping ? Express in radians the measure of the angle made by this position of the hands with the position at 12 o'clock.

(9) One foot of a horse is tethered to a stake by a rope 27 m. long. If the horse moves along the circumference of a circle always keeping the rope tight, find the distance through which the horse moves when the rope has rotated through an angle whose measure is 70°. (Take $\pi = 22/7$).

(10) A sector AOB of a circle with centre O and radius a is removed and the remainder is folded to form a right circular cone. Find the radius of the circular base of the cone if $m\ \angle AOB = 60°$.

(11) Fill in the blanks in the following :

(i) $\frac{\pi^c}{3} = \ldots 60 \ldots °$. (ii) $1° = \ldots 57'17'' \ldots$ radians. (iii) The arc AB of a circle of radius 6 cms. subtends an angle of 120° at the centre O. ∴ The length of the arc $AB = \ldots 4\pi \ldots$ cms. and the area of the sector $AOB = \ldots 12\pi \ldots$ sq. cms.

(iv) The length of the arc AB of a circle with centre O and radius 12 cms. is 4π cms. ∴ $m(\widehat{AOB}) = \ldots \frac{\pi}{3} \ldots$ radians and the area of the sector $AOB = \ldots 24\pi \ldots$ sq. cms.

(v) The measure of the acute angle between the hands of a clock at 2–15 p. m. is $\ldots 1.65° \ldots$.

(vi) The terminal arm of the angle of measure – 130° lies in the $\ldots 3rd \ldots$ quadrant if its initial arm is OX.

(12) State whether the following statements are true or false :

(i) For the rotation of measures $+ 40^\circ$ and $+ 400^\circ$ if the initial arm is the same, the terminal arm is the same.

(ii) A measure of rotation cannot be greater than 360°.

(iii) A rotation of measure -60° is the same as the rotation of measure $+ 300^\circ$.

(iv) The angle of measure 330° is coterminal with the angle of measure $-\frac{\pi^c}{6}$.

ANSWERS

(1) (i) $\frac{5\pi}{12}$, (ii) $\frac{13\pi}{12}$, (iii) $\frac{71\pi}{360}$, (iv) $-\frac{5\pi}{3}$, (v) $-\frac{55\pi}{12}$.

(2) (i) 240°, (ii) 945°, (iii) -15°, (iv) -252°.

(3) 40°, 60°, 80°; $\frac{2\pi}{9}$, $\frac{\pi}{3}$, $\frac{4\pi}{9}$. (4) $65^\circ 27'$.

(5) $m \angle A = \pi/36$, $m \angle B = \pi/180$. (6) 24°, 60°, 96°.

(7) $\frac{7\pi^c}{124}$. (8) $16\frac{4}{11}$ minutes past 3 o'clock.

(9) $33m$. (10) $r = 5a/6$.

(11) (i) 60, (ii) $\pi/180$, (iii) 4π, 12π, (iv) $\pi/3$, 24π, (v) $22\frac{1}{2}$ (vi) third.

(12) (i) and (iv) true; (ii) and (iii) false.

□ □ □

CHAPTER **2**

Circular (Trigonometrical) Functions and Identities

(1) Rectangular Co-ordinates : Students by this time are familiar with rectangular co-ordinates. In this system, the plane is divided into four quadrants by two intersecting perpendicular straight lines. The horizontal line $X'OX$ is called the x-axis, the vertical line $Y'OY$, the y-axis and the two together are called the co-ordinate axes. Their point of intersection O is called the **origin.** The distance from the y-axis measured parallel to the x-axis is called the x-co-ordinate or **abscissa** of the point. The distance from the x-axis measured parallel to the y-axis is called the y-co-ordinate or **ordinate** of the point. The two distances taken together are called the co-ordinates of the point and are represented by the ordered pair (x, y). Abscissae are positive when measured to the right of the y-axis and are negative when measured to the left. Ordinates are positive when measured above the x-axis, and negative if measured below. It should be noted here that the y-co-ordinates of all points on the x-axis are zero and the x-co-ordinates of all points on the y-axis are zero. It also follows that the origin O is (0,0). There is one-one correspondence from a set of points in a plane to a set of ordered pairs (x, y) where x, y are any real numbers.

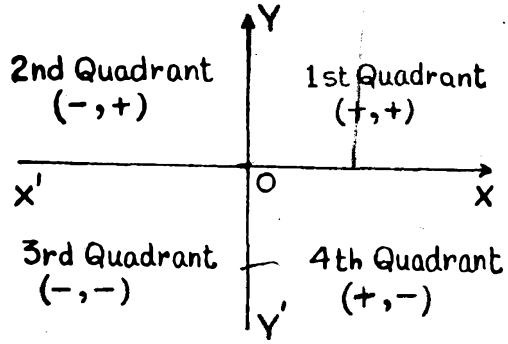


Fig. 2·1

Illustration : Consider Fig. 2·2.

Let $X'OX$ and $Y'OY$ be the axes of co-ordinates. Let P be a point in the first quadrant. Let seg $PM \perp X'OX$ and seg $PN \perp Y'OY$. Abscissa of $P = OM = 4$, and ordinate of P is $ON = 3$. Thus P has co-ordinates (4, 3).

Let P' be a point in the third quadrant. Let seg $P'M' \perp X'OX$ and seg $P'N' \perp Y'OY$. Abscissa of $P' = OM' = -3{\cdot}5$ and ordinate of $P' = ON' = -2{\cdot}5$. Thus P has co-ordinates $(-3{\cdot}5, -2{\cdot}5)$. In this way the co-ordinates of any point in a plane can be determined with reference to a rectangular frame of axes and, conversely, any point whose co-ordinates are given can be plotted on the plane.

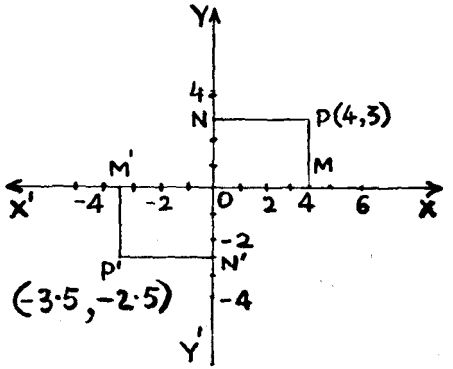


Fig. 2·2

(2) **Trigonometrical Ratios** : We shall now define the six trigonometrical ratios.

If a straight line, initially coincident with OX, revolves about the point O so as to generate an angle XOA then we say that $\angle XOA$ is in the standard position. Denote the measure of this angle XOA by α. Let $X'OX$ and $Y'OY$ be taken as the axes of co-ordinates.

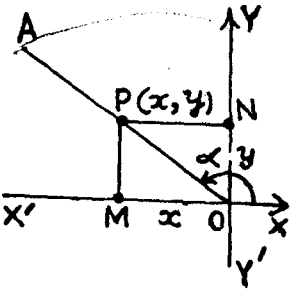


Fig. 2·3

On the line OA take any point P. Draw, from P, seg PM and seg PN perpendicular to the x-axis and the y-axis respectively and let the co-ordinates of P be (x, y). We shall denote the length of OP by r; (r will always be taken as positive.) We now also have $OM = x$ and $MP = y$. The signs of x and y will depend upon the quadrant in which the point P lies. With this notation we define the six trigonometrical ratios of any angle of measure α.

Definitions : We have

(i) cosine of α, briefly written as $\cos \alpha$, $= x/r$,

(ii) sine of α, briefly written as $\sin \alpha$, $= y/r$,

(iii) tangent of α, briefly written as $\tan \alpha$, $= y/x$, $x \neq 0$,

(iv) secant of α, briefly written as $\sec \alpha$, $= r/x$, $x \neq 0$,

(v) cosecant of α, briefly written as $\operatorname{cosec} \alpha$, $= r/y$, $y \neq 0$,

(vi) cotangent of α, briefly written as $\cot \alpha$, $= x/y$, $y \neq 0$.

It is evident from the definitions (iv), (v) and (vi) that sec α, cosec α and cot α (except when they are undefined) are just the reciprocals of cos α, sin α and tan α respectively, as defined in (i), (ii) and (iii). We thus have

$$\sec \alpha = \frac{1}{\cos \alpha}, \quad \operatorname{cosec} \alpha = \frac{1}{\sin \alpha} \quad \text{and} \quad \cot \alpha = \frac{1}{\tan \alpha}.$$

The following points about the above definitions should be noted.

Note 1 : When x or OM is zero, i. e. when the point P lies on the y-axis, ratios y/x and r/x are not defined; hence tan α and sec α are not defined when

$$\alpha = n\pi + \pi/2, \; n \in I.$$

Similarly when y, i.e. MP, is zero i. e. when the point P lies on the x-axis, the ratios x/y and r/y are not defined; hence cot α and cosec α are not defined when $\alpha = n\pi$, $n \in I$.

Note 2 : In the above definitions it is important to note that we have taken P as any point on the revolving line OA. This means that although the values of x, y, r depend on the position of the point P on OA (and obviously on the final position of OA), the ratios x/r, y/r, y/x (i.e. cos α, sin α, tan α) are independent of the position of P on OA. This can be clearly seen from figure 2·4. Here P and P' are two points on OA which is shown in its final position in the figure.

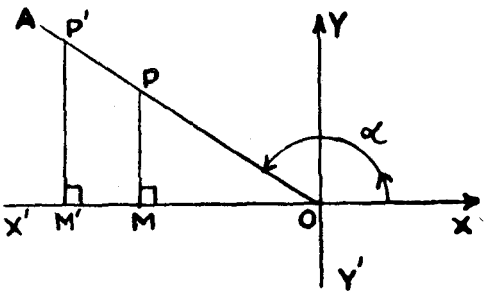


Fig. 2·4

From the similar triangles OPM, $OP'M'$ we have

$$\frac{OM}{OM'} = \frac{MP}{M'P'} = \frac{OP}{OP'};$$

$$\therefore \frac{OM}{OP} = \frac{OM'}{OP'}$$

and $$\frac{MP}{OP} = \frac{M'P'}{OP'}.$$

This shows that the ratios cos α and sin α and hence also tan α do not depend on the position of P on OA.

Note 3 : Since $\cos \alpha = x/r$ and $\sin \alpha = y/r$, we have $x = r \cos \alpha$ and $y = r \sin \alpha$. Hence the co-ordinates (x, y) of P can also be stated as $(r \cos \alpha, r \sin \alpha)$.

(3) **Unit Circle** : Since P can be taken as any point on the revolving line OA, we can as well take $OP = r = 1$ unit. Then the co-ordinates (x, y) of the point P become $(\cos \alpha, \sin \alpha)$. (See note 3 of article 2 above). This can be used to give alternative definitions of the trigonometrical ratios as follows.

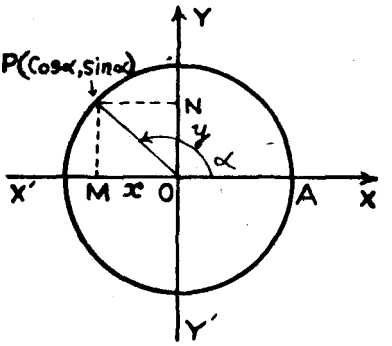


Fig. 2·5

Draw a unit circle, i. e. a circle with origin O as the centre and with one unit length as the radius. Let this circle cut OX at A and let P be any point on the circumference of the circle so that the measure of angle $AOP = \alpha$.

Then we define the x-co-ordinate of P, viz. OM, as $\cos \alpha$ and the y-co-ordinate of P, viz. MP, as $\sin \alpha$.

The remaining trigonometrical ratios are then defined as follows.

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \quad (\cos \alpha \neq 0),$$

$$\sec \alpha = \frac{1}{\cos \alpha}, \quad (\cos \alpha \neq 0), \quad \operatorname{cosec} \alpha = \frac{1}{\sin \alpha}, \quad (\sin \alpha \neq 0)$$

$$\text{and } \cot \alpha = \frac{1}{\tan \alpha}, \quad (\tan \alpha \neq 0).$$

(4) **Signs of the Trigonometrical Ratios** : From the definitions given in article 2, we have

$$\cos \alpha = \frac{x}{r}, \quad \sin \alpha = \frac{y}{r}, \quad \tan \alpha = \frac{y}{x}, \quad (x \neq 0). \qquad \ldots \quad (i)$$

Since r is always positive, the signs of these trigonometrical ratios depend upon the signs of x and y. Thus the signs depend upon the quadrant in which OP lies in its final position. We can, therefore, find the signs of x and y and determine the signs of these ratios.

(i) If the revolving line OP is in the first quadrant, so that $0 < \alpha < \pi/2$, then x, y are both positive; hence in this case all the three ratios viz. $\sin \alpha$, $\cos \alpha$, $\tan \alpha$ are positive.

(ii) If the revolving line OP is in the second quadrant, so that $\pi/2 < \alpha < \pi$, then x is negative and y is positive; hence in this case $\sin \alpha$ is positive and $\cos \alpha$ and $\tan \alpha$ are negative.

(iii) If the revolving line OP is in the third quadrant i e. $\pi < \alpha < 3\pi/2$. then x and y are both negative; hence in this case $\cos \alpha$ and $\sin \alpha$ are negative; but $\tan \alpha$ is positive.

(iv) If the revolving line OP is in the fourth quadrant i. e. $3\pi/2 < \alpha < 2\pi$ then x is positive and y is negative; hence in this case $\cos \alpha$ is positive; but $\sin \alpha$ and $\tan \alpha$ are both negative.

These results are given in the following table.

Quad.	$\sin \alpha$	$\cos \alpha$	$\tan \alpha$	Remarks
I	+	+	+	All the three ratios are + ve.
II	+	−	−	$\sin \alpha$ is + ve, the other two are − ve.
III	−	−	+	$\tan \alpha$ is + ve, the other two are − ve.
IV	−	+	−	$\cos \alpha$ is + ve, the other two are − ve.

This table of signs of the three ratios can be very easily remembered by associating the first letters of the **words** in '**All Silver Tea Cups**' with all ratios, $\sin \alpha$, $\tan \alpha$ and $\cos \alpha$ respectively and assigned a + ve sign in the serial order of the four quadrants.

If the line OP coincides with the x-axis, then the y-co-ordinate of P is zero and hence for such angles α, the sine is zero and the sign of the abscissa of P determines the sign of the cosine ratio. Similarly if the line OP coincides with the y-axis, then the x-co-ordinate of P is zero and hence for such angles α, the cosine is zero and the sign of the ordinate of P determines the sign of the sine ratio. If the signs of $\sin \alpha$ and $\cos \alpha$ are known, then the sign of $\tan \alpha$ can be determined since $\tan \alpha = \sin \alpha / \cos \alpha$, $\cos \alpha \neq 0$.

Since the remaining three trigonometrical ratios, namely, $\sec \alpha$, $\operatorname{cosec} \alpha$ and $\cot \alpha$ are the reciprocals of $\cos \alpha$, $\sin \alpha$ and $\tan \alpha$, the signs of the former are respectively the same as the signs of the latter.

Ex. Find the signs of $\cos \alpha$ and $\cot \alpha$ if $m(\alpha) = (-8\pi/3)^c$.

Here the measure of the angle α is given in radians.

We have $(8\pi/3)^c = \frac{8\pi}{3} \times \frac{180}{\pi}$ degrees $= 480°$.

Thus $m(\alpha) = (-8\pi/3)^c = -480° = -360° - 120°$.

Hence the revolving line OP is in the third quadrant; hence $\cos \alpha$ is negative; and $\cot \alpha$ is positive.

(5) **A special case of acute angle** : The definitions given in article 2 take a simpler form when the given angle α is acute i. e. when $0 < \alpha < \pi/2$. It should be noted that all the trigonometrical ratios of an acute angle are positive. (See Fig. 2·6). The definitions, as known to you, are

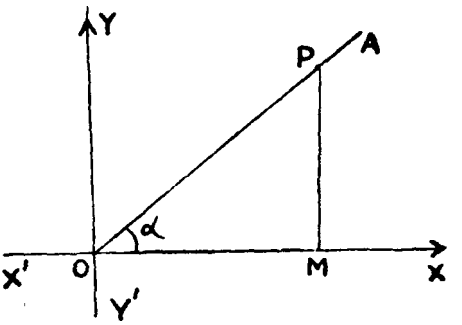


Fig. 2·6

$$\cos\alpha = \frac{\text{adj. side}}{\text{hyp.}}, \sin\alpha = \frac{\text{opp. side}}{\text{hyp.}}, \tan\alpha = \frac{\text{opp. side}}{\text{adj. side}},$$

$$\sec\alpha = \frac{\text{hyp.}}{\text{adj. side}}, \operatorname{cosec}\alpha = \frac{\text{hyp.}}{\text{opp. side}}, \cot\alpha = \frac{\text{adj. side}}{\text{opp. side}}.$$

(6) **The values of trigonometric ratios of special angles of measure**
(a) (i) 0, (ii) $\pi/2$, (iii) π and (iv) $3\pi/2$,
(b) $\pi/6$, (c) $\pi/4$ and $3\pi/4$, (d) $\pi/3$ and $2\pi/3$ radians.

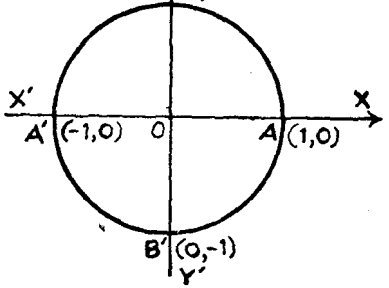


Fig. 2·7

Let the unit circle with the origin O as centre cut the x-axis in the points A' $(-1, 0)$, A $(1, 0)$ and the y-axis in the points B' $(0, -1)$ and B $(0, 1)$. The points A, B, A' and B' then represent the numbers 0, $\pi/2$, π and $3\pi/2$ respectively. These numbers are the measures of the corresponding angles in the circular measure.

Hence we get the following results by using the definitions of trigonometrical ratios.

(a) (i) Trigonometric ratios of the angle whose measure is zero.

$$\cos 0 = 1, \quad \sin 0 = 0 ; \tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0;$$

$$\cot 0 = \frac{\cos 0}{\sin 0} = \frac{1}{0} \text{ and so is not defined;}$$

$$\text{also } \sec 0 = \frac{1}{\cos 0} = 1 ; \operatorname{cosec} 0 = \frac{1}{\sin 0} = \frac{1}{0} \text{ and so is not defined.}$$

(a) (ii) Trigonometric ratios of the angle whose measure is $\pi/2$.

$\cos \pi/2 = 0$; $\sin \pi/2 = 1$.

$\tan \pi/2$ and $\sec \pi/2$ are not defined each being $1/0$;

$$\cot \pi/2 = \frac{\cos \pi/2}{\sin \pi/2} = 0/1 = 0; \text{ and } \operatorname{cosec} \pi/2 = \frac{1}{\sin \pi/2} = 1.$$

(a) (iii) Trigonometric ratios of the angle whose measure is π.

$$\cos \pi = -1, \quad \sin \pi = 0; \quad \tan \pi = \frac{\sin \pi}{\cos \pi} = \frac{0}{-1} = 0;$$

$$\sec \pi = \frac{1}{\cos \pi} = \frac{1}{-1} = -1;$$

$$\cot \pi = \frac{\cos \pi}{\sin \pi} = \frac{-1}{0} \text{ and so is not defined;}$$

$$\operatorname{cosec} \pi = \frac{1}{\sin \pi} = \frac{1}{0} \text{ and so is not defined.}$$

(a) (iv) Trigonometric ratios of the angle whose measure is $3\pi/2$.

$\cos 3\pi/2 = 0$, $\sin 3\pi/2 = -1$;

$$\tan 3\pi/2 = \frac{\sin 3\pi/2}{\cos 3\pi/2} = \frac{-1}{0} \text{ and so is not defined;}$$

$$\sec 3\pi/2 = \frac{1}{\cos 3\pi/2} = 1/0 \text{ and so is not defined;}$$

$$\operatorname{cosec} 3\pi/2 = \frac{1}{\sin 3\pi/2} = \frac{1}{-1} = -1; \cot 3\pi/2 = \frac{\cos 3\pi/2}{\sin 3\pi/2} = \frac{0}{-1} = 0.$$

(b) Trigonometric ratios of the angle whose measure is $\pi/6$.

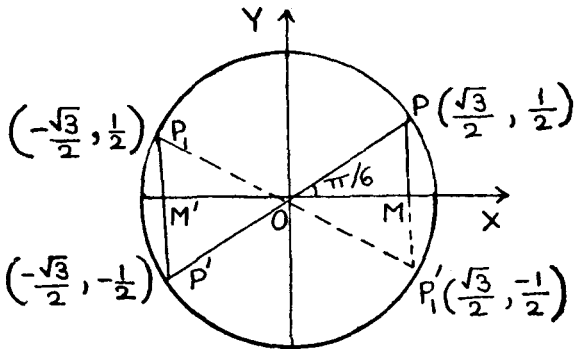


Fig. 2.8

Let the point P on the unit circle represent the number $\pi/6$. Let seg $PM \perp OX$.

From the right-angled triangle OMP in which $(\widehat{MOP}) = \pi/6 = 30°$, $MP = 1/2$ and $OM = \sqrt{1-(1/2)^2} = \sqrt{3}/2$.

$\therefore P = (\sqrt{3}/2,\ 1/2)$.

$\therefore \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$; $\tan\frac{\pi}{6} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$;

$\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$, $\operatorname{cosec}\pi/6 = 2$ and $\cot\pi/6 = \sqrt{3}$.

If P' is the point on the unit circle diametrically opposite to P then $P' = (-\sqrt{3}/2, -1/2)$ will represent the number $7\pi/6$ and hence the trigonometrical ratios of $7\pi/6$ can be written down.

If the point P_1 on the unit circle represents the number $5\pi/6$ and seg $P'P_1 \perp OX$ then P_1 is $(-\sqrt{3}/2,\ 1/2)$. Hence the trigonometrical ratios of $5\pi/6$ can be written down.

Thus $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$.

(c) Trigonometric ratios of Angles of measure $\pi/4$ and $3\pi/4$.

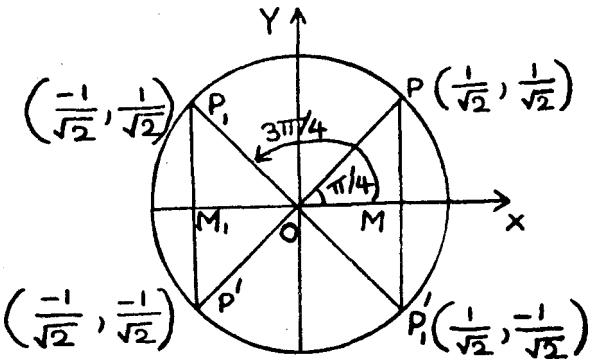


Fig. 2·9

Let the points P and P_1 represent the numbers $\pi/4$ and $3\pi/4$ on the unit circle. Let seg $PM \perp OX$, so that $m(\widehat{MOP}) = \pi/4$.

$\therefore$ from the isosceles right-angled triangle OMP we have $OM = MP = 1/\sqrt{2}$, and $P \equiv (1/\sqrt{2},\ 1/\sqrt{2})$.

Similarly, $P_1 M_1$ being $\perp OX$,

we have $OM_1 = M_1P_1 = 1/\sqrt{2}$.

$\because$ P_1 is in the second quadrant $\therefore$ $P_1 \equiv (-1/\sqrt{2}, 1/\sqrt{2})$.

$\therefore$ $\cos \pi/4 = 1/\sqrt{2}$, $\sin \pi/4 = 1/\sqrt{2}$, $\tan \pi/4 = 1$;

and $\sec \pi/4 = \sqrt{2}$, $\operatorname{cosec} \pi/4 = \sqrt{2}$ and $\cot \pi/4 = 1$.

Similarly we have,

$\cos (3\pi/4) = -1/\sqrt{2}$, $\sin (3\pi/4) = 1/\sqrt{2}$, $\tan (3\pi/4 =) -1$;

$\sec (3\pi/4) = -\sqrt{2}$, $\operatorname{cosec} (3\pi/4) = \sqrt{2}$, $\cot (3\pi/4) = -1$.

If P' and P'_1 be points on the unit circle diametrically opposite to P and P_1 respectively, they will represent the numbers $5\pi/4$ and $7\pi/4$ respectively and since $P' = (-1/\sqrt{2}, -1/\sqrt{2})$ and $P'_1 = (1/\sqrt{2}, -1/\sqrt{2})$ the trigonometrical ratios of $5\pi/4$ and $7\pi/4$ can be written down.

(d) Trigonometric ratios of Angles of measure $\pi/3$ and $2\pi/3$.

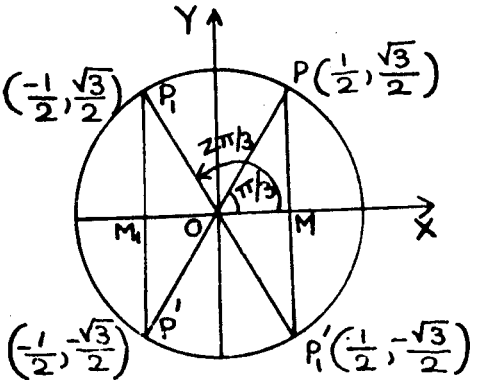


Fig. 2.10

Let the points P and P_1 on the unit circle represent the numbers $\pi/3$ and $2\pi/3$ respectively and let seg PM and seg $P_1M_1 \perp OX$.

Then from the right-angled triangle OMP in which $m(\widehat{OPM}) = \pi/6$ we get $OM = 1/2$ and $MP = \sqrt{1 - (1/2)^2} = \sqrt{3}/2$;

$\therefore$ $P = (1/2, \sqrt{3}/2)$

and similarly $P_1 = (-1/2, \sqrt{3}/2)$.

$\therefore$ $\cos \pi/3 = 1/2$, $\sin \pi/3 = \sqrt{3}/2$, $\tan \pi/3 = \sqrt{3}$;

$\sec \pi/3 = 2$, $\operatorname{cosec} \pi/3 = 2/\sqrt{3}$ and $\cot \pi/3 = 1/\sqrt{3}$.

Similarly, we have

$\cos (2\pi/3) = -1/2$, $\sin (2\pi/3) = \sqrt{3}/2$, $\tan 2\pi/3 = -\sqrt{3}$;

$\sec (2\pi/3) = -2$, $\operatorname{cosec} (2\pi/3) = 2/\sqrt{3}$ and $\cot 2\pi/3 = -1/\sqrt{3}$.

Considering the points P' and P'_1, on the unit circle, diametrically opposite to P and P_1 respectively, we can similarly obtain the trigonometrical ratios of $4\pi/3$ and $5\pi/3$.

The values of the trigonometrical ratios of some important angles can be remembered with the help of the following table.

α in degrees	0	30°	45°	60°	90°
α in radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\alpha$	$\sqrt{0/4}$ i. e. 0	$\sqrt{1/4}$ i. e. 1 2	$\sqrt{2/4}$ i. e. $1/\sqrt{2}$	$\sqrt{3/4}$ i. e. $\sqrt{3}/2$	$\sqrt{4/4}$ i. e. 1
$\cos\alpha$	$\sqrt{4/4}$ i. e. 1	$\sqrt{3/4}$ i. e. $\sqrt{3}/2$	$\sqrt{2/4}$ i. e. $1/\sqrt{2}$	$\sqrt{1/4}$ i. e. 1/2	$\sqrt{0/4}$ i. e. 0

In writing the table for $\sin\alpha$, it may be noted that the numbers 0, 1, 2, 3, 4 are taken in the serial order; each is divided by 4 and then the positive square root is taken.

In writing the table for $\cos\alpha$, it may be noted that the numbers 0, 1, 2, 3, 4 are taken in the reverse order; each is divided by 4 and then the positive square root is taken.

Once the values of $\sin\alpha$ and $\cos\alpha$ are known, the values of the other ratios can be obtained since all the other ratios can be expressed in terms of $\sin\alpha$ and $\cos\alpha$.

It may be remembered that the values of $\sin(n\pi/2)$ and $\cos(n\pi/2)$ where n is any integer can always be obtained by observing the ordinate and abscissa of the point representing the angle $n\pi/2$ on the unit circle.

(7) Fundamental Identities : We have seen that the trigonometrical ratios of an angle depend only upon the sign and magnitude of the angle. The evaluation of the trigonometrical ratios of angles which are not acute and positive is much simplified with the help of (i) the formulae for allied angles and (ii) the relations between the ratios. We shall study the former in Chapter 6 and consider the latter in this article.

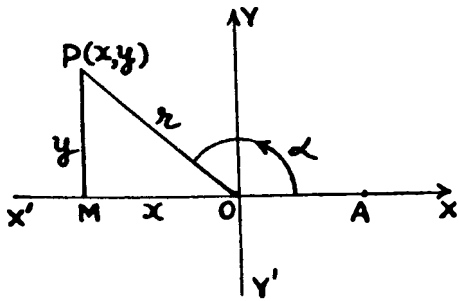


Fig. 2·11

Let $X'OX$ and $Y'OY$ be the x and $y-$axes; let the revolving line OP start from the position OA, which lies along OX, and revolve through an angle AOP, say, of measure equal to α. Let the co-ordinates of P be (x, y) and $OP = r$ as in Fig. 2·11. We also have

$\cos \alpha = \frac{x}{r}$, $\sin \alpha = \frac{y}{r}$ and $\tan \alpha = \frac{y}{x}$, $(x \neq 0)$.

Hence $\tan \alpha = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin \alpha}{\cos \alpha}$. ... (1)

From the right angled triangle OPM, we have

$OM^2 + MP^2 = OP^2$, i. e. $x^2 + y^2 = r^2$. ... (A)

Dividing by r^2, we get $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$,

i. e. $(\cos \alpha)^2 + (\sin \alpha)^2 = 1$, i. e. $\cos^2 \alpha + \sin^2 \alpha = 1$. ... (2)

Dividing (A) by x^2 $(\neq 0)$, we get

$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$, i. e. $1 + \tan^2 \alpha = \sec^2 \alpha$. ... (3)

Dividing (A) by y^2 $(\neq 0)$, we get similarly

$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2}$, i. e. $\cot^2 \alpha + 1 = \operatorname{cosec}^2 \alpha$. ... (4)

Alternative Method : Let P lie on the unit circle with its centre at O. With the usual axes, let

$O = (0,0)$, $P = (x, y)$ and $m(\angle AOP) = \alpha$.

By the distance formula (see Chapter 3), We have

$x^2 + y^2 = OP^2$ ... (i)

But $OP = 1$, since P lies on the unit circle. ... (ii)

$\therefore$ from (i) and (ii), we have $x^2 + y^2 = 1$. ... (iii)

But by definitions, $x = \cos \alpha$ and $y = \sin \alpha$.

$\therefore$ from (iii), we have $\cos^2 \alpha + \sin^2 \alpha = 1$. ... (2)

We can obtain formulae (3) and (4) from (2); thus dividing (2) by $\cos^2 \alpha$, we get, if $\cos \alpha \neq 0$,

$1 + \frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{1}{\cos^2 \alpha}$, i e., $1 + \tan^2 \alpha = \sec^2 \alpha$. .. (3)

Similarly dividing (2) by $\sin^2 \alpha$ we get, if $\sin \alpha \neq 0$

$\frac{\cos^2 \alpha}{\sin^2 \alpha} + 1 = \frac{1}{\sin^2 \alpha}$, i. e. $\cot^2 \alpha + 1 = \operatorname{cosec}^2 \alpha$. ... (4)

Note 1: The above relations (2) to (4) are important. They show that if one of the six trigonometrical ratios of an angle is known, all the other trigonometrical ratios of that angle can be calculated numerically. This is shown in the examples of the next article.

Note 2: It is customary to write $(\cos \propto)^2$ as $\cos^2 \propto$, $(\tan \propto)^2$ as $\tan^2 \propto$ etc. The interpretation of $\sin^2 \propto$, $\operatorname{cosec}^2 \propto$, $\cot^2 \propto$ etc. is similar.

Note 3 : An equation in $a, b, x, \ldots$ is called an identity if it is true for all values of $a, b, x, \ldots$ in the relation of equality.

Thus $a^2 - b^2 = (a + b)(a - b)$ is an identity as it holds for all values of a and b, but $x^2 - 3x + 2 = 0$ is an equation as it holds only when $x = 1$ or $x = 2$.

The relation (2) holds for all values of $\propto$; as such it is called a trigonometrical identity. The relations (3) and (4) also hold for all values of $\propto$ except those values of $\propto$ for which the trigonometrical ratios involved in them are undefined. However, it is customary to call all the relations (2) to (4) as trigonometrical identities.

Illustrative Examples :

Ex. 1. If $\cot \propto = -3$ and $3\pi/2 < \propto < 2\pi$ find all the trigonometrical ratios of $\propto$.

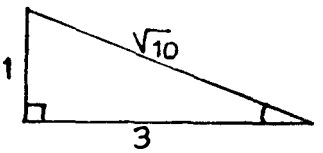


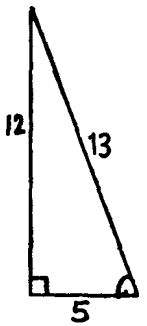
Fig. 2·12

$\propto$ lies in the fourth quadrant. This fact will determine the signs of the trigonometrical ratios whose magnitudes can be quickly written down by using the right angled Δ shown in Fig. 2·12 (note that $1^2 + 3^2 = 10$).

$\therefore \sin \propto = -1/\sqrt{10}$, $\cos \propto = 3/\sqrt{10}$, $\tan \propto = -1/3$, etc.

Ex. 2. Given $\sec \theta = -13/5$, and $\pi < \theta < 3\pi/2$ find the value of

$$\frac{\operatorname{cosec} \theta - \cot \theta}{\sec \theta + \tan \theta}.$$



We note that $13^2 - 5^2 = 12^2$. Using the fact that in this case θ lies in the third quadrant and using the Fig. 2·13 we have $\operatorname{cosec} \theta = -13/12$, $\cot \theta = 5/12$, $\tan \theta = 12/5$.

$$\therefore \frac{\operatorname{cosec} \theta - \cot \theta}{\sec \theta + \tan \theta}$$

$$= \frac{(-13/12) - 5/12}{(-13/5) + 12/5} = \frac{-18/12}{-1/5} = 15/2.$$

Fig. 2·13

Ex. 3. Prove that $\sin^6 x + \cos^6 x = 1 - 3 \sin^2 x \cos^2 x$.

We have L. H. S. $= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)$

$= 1 \times [(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x]$

$= 1 - 3 \sin^2 x \cos^2 x$ = R. H. S.

Ex. 4. Prove that $\tan x + \cot x = \sec x \operatorname{cosec} x$.

We have L. H. S. $= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x}.$$

R. H. S. $= \sec x \operatorname{cosec} x = \dfrac{1}{\cos x} \cdot \dfrac{1}{\sin x}$:

$\therefore$ L. H. S. = R. H. S.

Ex. 5. Prove that

$$(1 - \tan y)^2 + (1 - \cot y)^2 = (\sec y - \operatorname{cosec} y)^2.$$

We have L. H. S. $= (1 - 2 \tan y + \tan^2 y) + (1 - 2 \cot y + \cot^2 y)$

$= (1 + \tan^2 y) + (1 + \cot^2 y) - 2 (\tan y + \cot y)$

$= \sec^2 y + \operatorname{cosec}^2 y - 2 \sec y \operatorname{cosec} y$ (see Ex. 4 above)

$= (\sec y - \operatorname{cosec} y)^2 =$ R. H. S.

Ex. 6. Prove that $\dfrac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \sec \theta + \tan \theta$.

It is enough to prove that

$(\sec \theta + \tan \theta)(\tan \theta - \sec \theta + 1) = \tan \theta + \sec \theta - 1$.

Now $(\sec \theta + \tan \theta)(\tan \theta - \sec \theta + 1)$

$= \sec \theta \tan \theta - \sec^2 \theta + \sec \theta + \tan^2 \theta - \tan \theta \sec \theta + \tan \theta$

$= (-\sec^2 \theta + \tan^2 \theta) + \sec \theta + \tan \theta$

$= \sec \theta + \tan \theta - 1$, $\because \sec^2 \theta = 1 + \tan^2 \theta$.

$\therefore \quad \dfrac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \sec \theta + \tan \theta$.

Ex. 7. Prove that $\sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta}} = \sec \theta - \tan \theta$.

We have L. H. S. $= \sqrt{\dfrac{(1 - \sin \theta)^2}{(1 + \sin \theta)(1 - \sin \theta)}}$

$$= \sqrt{\frac{(1 - \sin \theta)^2}{1 - \sin^2 \theta}} = \frac{1 - \sin \theta}{\cos \theta}$$

$$= \sec \theta - \tan \theta = \text{R. H. S.}$$

Ex. 8. Prove that

$$\frac{1 + \operatorname{cosec} \theta + \cot \theta}{1 + \operatorname{cosec} \theta - \cot \theta} = \frac{\operatorname{cosec} \theta + \cot \theta - 1}{\cot \theta - \operatorname{cosec} \theta + 1}.$$

It is enough to prove, by componendo and dividendo, that

$$\frac{2 (1 + \operatorname{cosec} \theta)}{2 \cot \theta} = \frac{2 \cot \theta}{2 (\operatorname{cosec} \theta - 1)},$$

i. e. $(1 + \operatorname{cosec} \theta)(\operatorname{cosec} \theta - 1) = \cot^2 \theta$,

i. e. $\operatorname{cosec}^2 \theta - 1 = \cot^2 \theta$, i. e. $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$, which is an identity. Hence the result follows.

Ex. 9. Eliminate θ from

(i) $x = a \operatorname{cosec} \theta + b \cot \theta$, $y = a \operatorname{cosec} \theta - b \cot \theta$,

(ii) $x = a \cos^3 \theta$, $y = b \sin^3 \theta$.

(i) We have $x + y = 2a \operatorname{cosec} \theta$, $x - y = 2b \cot \theta$.

$$\therefore \left(\frac{x+y}{2a}\right)^2 - \left(\frac{x-y}{2b}\right)^2 = \operatorname{cosec}^2\theta - \cot^2\theta = 1.$$

$$\therefore \frac{(x+y)^2}{a^2} - \frac{(x-y)^2}{b^2} = 4.$$

(ii) We have $\frac{x}{a} = \cos^3\theta$, $\frac{y}{b} = \sin^3\theta$.

$$\therefore \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1, \text{ as } \cos^2\theta + \sin^2\theta = 1.$$

Ex. 10. Prove that

(i) $0 < \frac{\sin\theta}{\theta} < 1$, $0 < \frac{\theta}{\tan\theta} < 1$, if $0 < \theta < \frac{\pi}{2}$.

(ii) $\sqrt{1-\theta^2} < \frac{\sin\theta}{\theta} < 1$, $\frac{1}{\sqrt{1-\theta^2}} > \frac{\tan\theta}{\theta} > 1$ if $0 < \theta < 1$.

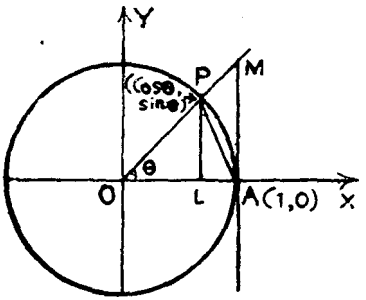


Fig. 2·14

Let P represent the number θ on the unit circle and $A = (1, 0)$ so that $m(\widehat{AOP}) = \theta$, where $0 < \theta < \pi/2$, the point P being in the first quadrant. Draw seg $PL \perp OA$ and let the tangent to the circle at A meet OP produced in M.

Now area of ($\triangle OAP$) < area of (sector OAP) < area of ($\triangle OAM$);

$$\therefore \frac{1}{2} \times 1 \times \sin\theta < \frac{1}{2} \times 1^2 \times \theta < \frac{1}{2} \times 1 \times \tan\theta.$$

$$\therefore \sin\theta < \theta < \tan\theta.$$

$$\therefore 0 < \frac{\sin\theta}{\theta} < 1 \text{ and } 0 < \frac{\theta}{\tan\theta} < 1, \text{ if } 0 < \theta < \pi/2. \quad \ldots (i)$$

Let us now assume that $0 < \theta < 1$.

Since $\theta > \sin\theta$, $\therefore \cos\theta = \sqrt{1-\sin^2\theta} > \sqrt{1-\theta^2}$, ... (ii)

also since $\tan\theta > \theta$, $\therefore \sin\theta > \theta\cos\theta$. ... (iii)

From (ii) and (iii) we get $\sin\theta\cos\theta > \theta\cos\theta\sqrt{1-\theta^2}$.

$$\therefore \sin\theta > \theta\sqrt{1-\theta^2}.$$

$$\therefore \sqrt{1-\theta^2} < \frac{\sin\theta}{\theta} < 1, \text{ if } 0 < \theta < 1. \quad \ldots (iv)$$

Again since $\sin\theta < \theta$ and $\frac{1}{\cos\theta} < \frac{1}{\sqrt{1-\theta^2}}$,

we have by multiplication $\frac{\sin\theta}{\cos\theta} < \frac{\theta}{\sqrt{1-\theta^2}}$.

$\therefore \frac{1}{\sqrt{1-\theta^2}} > \frac{\tan\theta}{\theta} > 1$, if $0 < \theta < 1$. ... (v)

The results proved here, will be used to prove that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1.$$

(8) **Bounds of sin $\propto$ and cos $\propto$.**

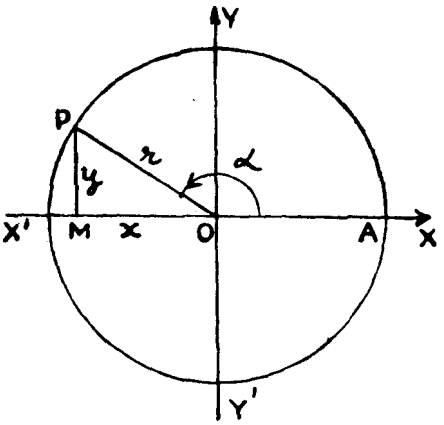


Fig. 2.15

We have

$\sin^2\propto + \cos^2\propto = 1$.

Each term on the L. H. S. is non-negative.

$\therefore \sin^2\propto \leqslant 1$

and $\cos^2\propto \leqslant 1$.

$\therefore \sin\propto$ and $\cos\propto$ are each numerically $\leqslant 1$;

i. e. $-1 \leqslant \sin\propto \leqslant 1$;

$-1 \leqslant \cos\propto \leqslant 1$.

The results can be verified by taking $r = 1$ in Fig. 2.15.

Note 1 : If $\sin\propto = \pm 1$, then $\cos\propto = 0$; and if $\cos\propto = \pm 1$, then $\sin\propto = 0$.

Note 2 : -1 and $+1$ are respectively called the lower bound and the upper bound of both $\sin\propto$ and $\cos\propto$. Later, it will be seen that $\tan\propto$ is not bounded.

(9) **Domain, Range Periodicity and Graphs of Circular Functions :**

(1) **Let $f(\propto) = \sin\propto$.** Since $\sin\propto$ is defined for every real value of its **domain**, its domain is the set of real numbers R. Since for any $\propto$, $\sin\propto$ lies between -1 and 1 (both values included), the **range** of $\sin\propto$ is the closed interval $[-1, 1]$. Since only one and the same point on the unit circle represents alone the angles $\propto + 2n\pi$, where n is any integer, we have $\sin(\propto + 2n\pi) = \sin\propto$. Hence $\sin\propto$ is a **periodic** function and the smallest positive value of $2n\pi$, namely, 2π is the period of the sine function.

Graph of $y = \sin x$: We construct the following table giving the corresponding values of x and y :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
y	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{-1}{2}$	-1	$-\frac{1}{2}$	0

The points, given in the above table, when joined successively by means of a smooth curve give us the graph of $y = \sin x$, in the interval $[\,0, 2\pi\,]$.

It can be seen that the graph in $[\,-2\pi, 0\,]$ is similar to the graph in $[\,0, 2\pi\,]$ due to the periodic nature of $\sin x$. The graph is shown in Fig. 2·16.

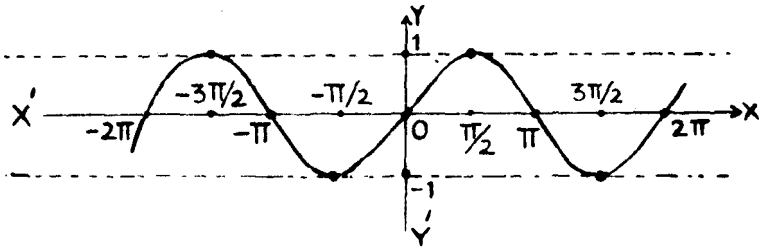


Fig. 2·16

(2) Let $f(\alpha) = \cos\alpha$. As in the case of the sine function the **domain** of $\cos\alpha$ is the set R and its **range** is the closed interval $[\,-1, 1\,]$. It can also be seen that $\cos\alpha$ is a periodic function with the period 2π, as has been seen above in (1) in the case of sine function.

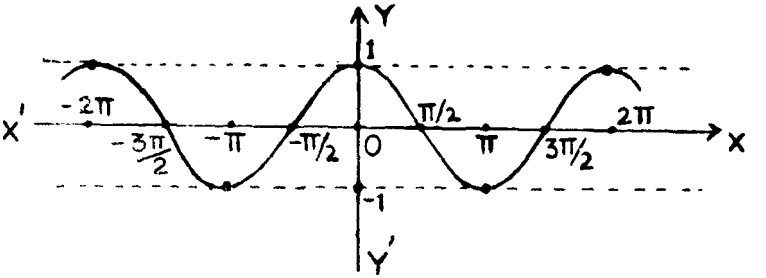


Fig. 2·17

The Graph of $y = \cos x$: We construct the following table giving the corresponding values of x and y :

x	0	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$3\pi/2$	$5\pi/3$	2π
y	1	1/2	0	$-1/2$	-1	$-1/2$	0	1/2	1

The points given in the above table, when joined successively by means of a smooth curve, give us the graph of $y = \cos x$ in the interval $[0, 2\pi]$. It can be seen that the graph in $[-2\pi, 0]$ is similar to the graph in $[0, 2\pi]$ due to the periodic nature of $\cos x$. The graph is shown in Fig. 2·17.

(3) Let $f(\propto) = \tan \propto$. We have with the usual notation the definition $\tan \propto = y/x$. Hence $\tan \propto$ is not defined when $x = 0$ i. e. when $\propto$ is a positive or negative odd multiple of $\pi/2$. Hence the **domain** of the tangent function is the set of real numbers R except the numbers $(2n+1)\pi/2$ where n is any integer. Since $\tan \propto$ can assume any real value, the **range** of the tangent function is the set R. Since only the same point or the pair of diametrically opposite points on the unit circle represent alone the angles $\propto$ and $\propto + n\pi$ where n is any integer and their tangent functions have the same value (viz. y/x or $-y/-x$) we have $\tan(\propto + n\pi) = \tan \propto$. Hence $\tan \propto$ is a **periodic** function and the smallest positive value of $n\pi$, namely, π is its period.

The Graph of $y = \tan x$: We observe that

(i) $\tan x = 0$ when $x = 0, \pm \pi, \pm 2\pi, \ldots, \pm n\pi, \ldots$

(ii) $\tan x$ is not defined when $x = (2n+1)\pi/2$ where n is any integer.

(iii) We prepare the following table for a few ordered pairs (x, y) :—

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\frac{\pi}{2}-$	$\frac{\pi}{2}+$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
y	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\infty$	$-\sqrt{3}$	-1	$-1/\sqrt{3}$	0

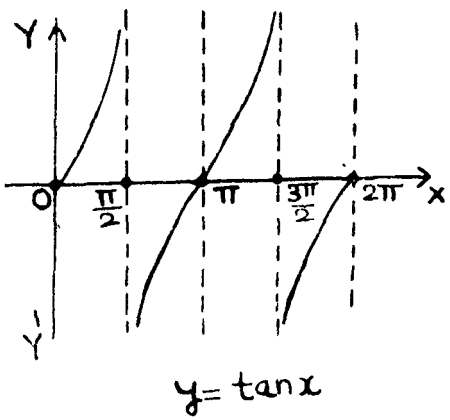


Fig. 2·18

(iv) In the semi-closed interval $\left[0, \frac{\pi}{2}\right)$, $\tan x$ is + ve and increases from zero and tends to $+\infty$.

In the semi-closed interval $\left(\frac{\pi}{2}, \pi\right]$, $\tan x$ is − ve and increases from $-\infty$ to 0.

(v) $\tan x$ is periodic with the period π.

The points given by the ordered pairs in the above table, when plotted and joined successively by means of a smooth curve, will give the graph of $y = \tan x$ as shown in Fig. 2·18. Due to the periodic nature of tan x its graph will repeat itself at intervals of π.

(4) Let f (α) = cosec α. We have cosec α = 1/sin α. Hence cosec α is not defined when sin $\alpha = 0$, i. e. when $\alpha = n\pi$ where n is any integer. Hence the **domain** of cosec x is the set of real numbers R except the numbers $n\pi$ where n is any integer.

We have $-1 \leqslant \sin \alpha \leqslant 1$.

$\therefore \quad -1 \leqslant \sin \alpha$ or $\sin \alpha \leqslant 1$.

$\therefore$ cosec $\alpha \leqslant -1$ or cosec $\alpha \geqslant 1$.

$\therefore$ the **range** of the cosec function consists of the two sets $(-\infty, -1]$ and $[1, \infty)$.

Cosec x, being the reciprocal of sin x, the cosec function, like the sine function, is **periodic** with the same period 2π.

The Graph of y = cosec x : We observe that (See Fig. 2·19)

(i) cosec x is not defined when $x = n\pi$, $n \in I$.

(ii) Since sin $x = 1$ when

$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \quad \ldots \quad \ldots$

and sin $x = -1$ when

$x = \frac{3\pi}{2}, \frac{7\pi}{2}, \quad \ldots \quad \ldots$

$\therefore$ cosec $x = 1$ and -1 for $x = \pi/2, 5\pi/2, 9\pi/2, \ldots$ and $x = 3\pi/2, 7\pi/2, \ldots$ respectively.

(iii) cosec $x \leqslant -1$ and cosec $x \geqslant 1$.

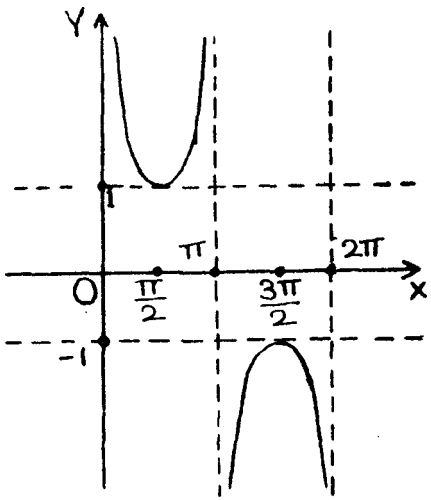


Fig. 2·19

(iv) We prepare the following table for a few ordered pairs (x, y) :

x	0+	$\pi/6$	$\pi/2$	$5\pi/6$	$\pi-$	$\pi+$	$3\pi/2$	$2\pi-$
y	∞	2	1	2	∞	$-\infty$	-1	$-\infty$

(v) In the semi-closed interval $(0, \pi/2]$, cosec x decreases from ∞ to 1 and in the semi-closed interval $[\pi/2, \pi)$, cosec x increases from 1 to ∞. In the semi-closed interval $(\pi, 3\pi/2]$, cosec x increases from $-\infty$ to -1 and in the semi-closed interval $[3\pi/2, 2\pi)$ cosec x decreases from -1 to $-\infty$.

(vi) cosec x is periodic with the period 2π.

The points, given by the ordered pairs in the above table, when plotted and joined successively by means of asmooth curve will give the graph of $y = \text{cosec } x$. Due to the periodic nature of cosec x its graph will repeat itself at intervals of 2π.

(5) Let f $(\alpha) = \sec \alpha$. We have $\sec \alpha = \dfrac{1}{\cos \alpha}$. Hence sec α is not defined when $\cos \alpha = 0$, i. e. when $\alpha = (2n + 1)\,\pi/2$, $n \in I$.

Hence the domain of sec α is the set of real numbrs R except the numbers $(2n + 1)\,\pi/2$ where $n \in I$.

Since $-1 \leqslant \cos \alpha \leqslant 1$, i. e. $\therefore \sec \alpha \leqslant -1$ or $\sec \alpha \geqslant 1$.

$\therefore$ The range of the sec function consists of the two sets $(-\infty, -1]$ and $[1, \infty)$.

Sec α, being the reciprocal of cos α, the sec function, like the cosine function, is **periodic** with the same period 2π.

The Graph of $y = \sec x$: We observe that (see Fig. 2·20)

(i) since $\cos x = 0$ when $x = (2n + 1)\,\pi/2$, $n \in I$,

$\therefore$ sec x is not defined for $x = (2n + 1)\,\pi/2$, $n \in I$,

(ii) since $\cos x = 1$, when $x = 2n\pi$, $n \in I$.

$\therefore \sec x = 1$, for $x = 2n\pi$, $n \in I$.

Similarly sec $x = -1$ when $x = (2n + 1)\,\pi$, $n \in I$.

(iii) Since $-1 \leqslant \cos x \leqslant 1$, $\therefore \sec x \leqslant -1$ or $\sec x \geqslant 1$.

(iv) We now prepare the following table for a few ordered pairs (x, y).

x	0	$\pi/3$	$\frac{\pi}{2}-$	$\frac{\pi}{2}+$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-$	$\frac{3\pi}{2}+$	$\frac{5\pi}{3}$	2π
y	1	2	∞	$-\infty$	-2	-1	-2	$-\infty$	∞	2	1

(v) In the semi-closed interval $[\,0, \pi/2\,)$, sec x increases from 1 to ∞. In the semi-closed interval $(\,\pi/2, \pi\,]$, sec x increases from $-\infty$ to -1. In the semi-closed interval $[\,\pi, 3\pi/2\,)$ sec x decreases from -1 to $-\infty$ and in $(\,3\pi/2, 2\pi\,]$ sec x decreases from ∞ to 1.

(vi) sec x is periodic with the period 2π.

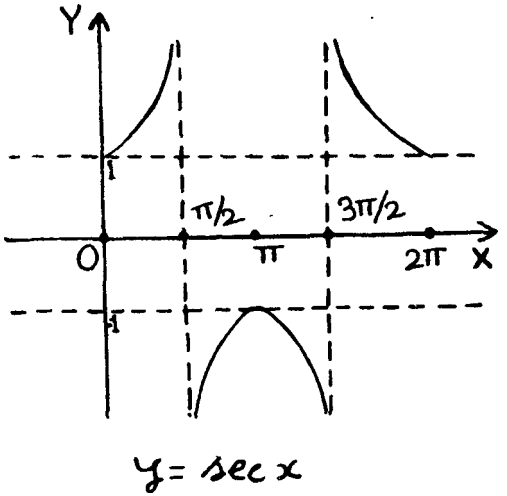


Fig 2·20

The points, given by the ordered pairs in the above table, when plotted and joined successively by means of a smooth curve will give the graph of $y = \sec x$. Due to the periodic nature of sec x, its graph will repeat itself at intervals of 2π.

(6) Let $f(\alpha) = \cot \alpha$. We have $\cot \alpha = 1/\tan \alpha$. Hence cot α is not defined when $\tan \alpha = 0$ i. e. when $\alpha = n\pi$, $n \in I$. Hence the **domain** of cot α is the set of real numbers R except the numbers $n\pi$, $n \in I$. Since cot α can assume any real value, the range of the cotangent function is the set R. Since cot α is the reciprocal of tan α, the cotangent function, like the tangent function, is **periodic** with the same period π.

Graph of $y = \cot x$: We observe that (See Fig. 2·21)

(i) $\tan x = 0$ when $x = n\pi$, $n \in I$.

$\therefore$ cot x is not defined when $x = n\pi$, $n \in I$;

(ii) tan x is not defined at $x = (2n + 1)\,\pi/2$, $n \in I$ and cot $x = 0$ at $x = (2n + 1)\,\pi/2$, $n \in I$.

(iii) We now prepare the following table for a few ordered pairs (x, y)

x	$0+$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$\pi-$	$\pi+$
y	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0	$-1/\sqrt{3}$	-1	$-\sqrt{3}$	$-\infty$	∞

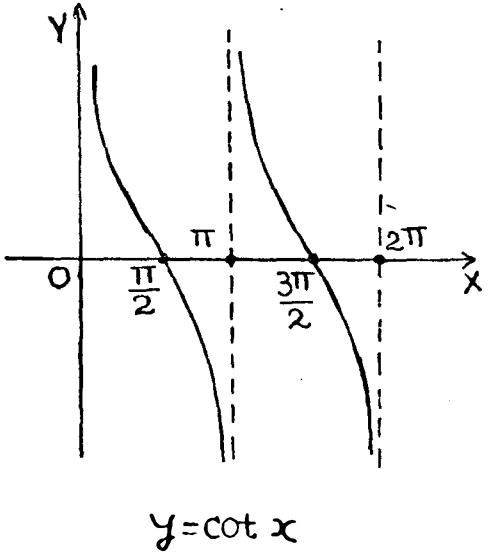


Fig. 2·21

(iv) In the semi-closed interval $(0, \pi/2]$, cot x is positive and decreases from ∞ to 0. In the semi closed interval $[\pi/2, \pi)$, cot x is negative and decreases from 0 to $-\infty$.

(v) cot x is periodic with the period π.

The points, given by the ordered pairs in the above table, when plotted and joined successively by means of a smooth curve will give the graph of $y = \cot x$. Due to the periodic nature of cot x, the graph will repeat itself at intervals of π.

Illustrative Examples :

Ex 1. Show that tan $3x$ is a periodic function and find its period.

We have tan $3x$ = tan $(3x + \pi)$ = tan $3(x + \pi/3)$.

$\therefore$ tan $3x$ is periodic with the period $\pi/3$.

Ex. 2. Find the periods of

(i) cos $(x/3)^\circ$, (ii) sin $8x$ + sin $2x$.

(i) cos $(x/3)^\circ$ = cos $(\frac{1}{3}x^\circ + 360^\circ)$ = cos $\frac{1}{3}(x^\circ + 1080^\circ)$.

$\therefore$ the period = 1080°.

(ii) The period of sin $8x$ is $2\pi/8$ and the period of sin $2x$ is $2\pi/2$. Obviously the period of the given function is $2\pi/2$ i. e. π, since the G. C. D. of 8 and 2 is 2. The period of (sin $8x$ + sin $2x$) is, therefore, also π.

Ex. 3. Construct a sine function whose period is 5.

Let the function be sin kx.

Now sin kx = sin $(kx + 2\pi)$ = sin $k(x + 2\pi/k)$.

Since the period is given as 5, we have $2\pi/k = 5$. $\therefore$ $k = 2\pi/5$.

$\therefore$ the required function is sin $(2\pi x/5)$.

Ex. 4. Examine whether $\sin(x^2)$ is a periodic function with period 2π.

We have $\sin(x^2) = \sin(x^2 + 2\pi)$.

But $\sin(x^2 + 2\pi) \neq \sin[(x + p)^2]$ for any non-zero value of p, independent of x. $\therefore$ $\sin(x^2)$ is not periodic.

EXERCISE 2

(1) Find from the definitions, the trigonometrical ratios of
(i) $\sin 720^\circ$ (ii) $\cos 3\pi$, (iii) $\cos 12\pi$ and (iv) $\sin 990^\circ$.

(2) If $\cos\alpha = -3/5$ and $\pi < \alpha < 3\pi/2$, find all the trigonometrical ratios of α.

(3) Find $\sin\alpha$ and $\cos\alpha$ in terms of k if $\tan\alpha = k$, a negative real number and $\pi/2 < \alpha < \pi$.

(4) If $\sin\alpha = 5/13$ and $\pi/2 < \alpha < \pi$, find the value of

$$\frac{\sec\alpha - \tan\alpha}{\operatorname{cosec}\alpha + \cot\alpha}.$$

(5) If $\sec x = 3$ and $3\pi/2 < x < 2\pi$, show that
$\sin x = -2\sqrt{2}/3$ and $\tan x = -2\sqrt{2}$.

(6) Find the signs of the ratios :
(i) $\sin(3\pi/4)$, (ii) $\cos 2^c$, (iii) $\cos 3\pi$, (iv) $\tan 200$.
(v) $\operatorname{cosec}(1300^\circ)$, (vi) $\sec(-400^\circ)$, (vii) $\cot(4^c)$.

Prove that (Ex. 7 to Ex. 19)

(7) $\dfrac{\sin x\cos y - \cos x\sin y}{\sin x\cos y + \cos x\sin y} = \dfrac{\tan x - \tan y}{\tan x + \tan y}$.

(8) $\cos^4\alpha - \sin^4\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$.

(9) $\dfrac{\cot x - \tan x}{\cot x + \tan x} = 1 - 2\sin^2 x$.

(10) $\dfrac{\cos x}{1 - \sin x} + \dfrac{1 - \sin x}{\cos x} = 2\sec x$.

(11) $(\sin x + \operatorname{cosec} x)^2 + (\cos x + \sec x)^2 = \tan^2 x + \cot^2 x + 7$.

(12) $\sec\theta\operatorname{cosec}\theta - 2\cos\theta\operatorname{cosec}\theta = \tan\theta - \cot\theta$.

(13) $\sin^6 y - \cos^6 y = (\sin^2 y - \cos^2 y)(1 - 2\sin^2 y\cos^2 y)$

(14) $\dfrac{1}{\sec x + 1} + \dfrac{1}{\sec x - 1} = 2\operatorname{cosec} x\cot x$.

(15) $\sec^6 x - \tan^6 x = 1 + 3\tan^2 x + 3\tan^4 x$.

(16) $1 - \tan^2 x + \tan^4 x = \cos^2 x(1 + \tan^6 x)$.

(17) $a^2 + b^2 = c^2 + d^2$,
if $a = c\sin x - d\cos x$ and $b = c\cos x + d\sin x$.

(18) $3\cot^2(\pi/3^c) + 2\cot^2(\pi/4) + \cot^2(\pi/6) = 2\tan^2(\pi/3)$.

(19) $\cos x = 4/5$, if $\cot x + \operatorname{cosec} x = 3$.
[**Hint** : $\operatorname{cosec}^2 x - \cot^2 x = 1$].

(20) State giving reasons whether the following statements are true or false :—

(i) $\sin x = -1.5$, (ii) $\cos x = a + 1/a$, $a \neq 0$,

(iii) $\sin^2 x - 3 \sin x + 2 = 0$, (iv) $\cos^2 x - 5 \cos x + 6 = 0$.

(21) State with reasons whether the following statements are true or false; and if true, state the conditions under which they are true :—

(i) $\sec^2 \theta = 2xy / (x^2 + y^2)$, $x, y \in R$, $x^2 + y^2 \neq 0$.

(ii) $\cos \theta = (x + y)^2/(4xy)$, $x, y \in R$, $x \neq y$, $x, y \neq 0$.

(iii) $\sin \theta = x + 1/x$, $x \in R$, $x \neq 0$.

(iv) $2 \cos \theta = x + 1/x$, $x \neq 0$.

(v) $3 \sin \theta + 5 \cos \theta = 9$.

(vi) $\sin^2 \theta - \cos^2 \theta > 0$, $0 < \theta < \pi/4$.

(vii) $\sin \theta - \cos \theta > 0$, $0 < \theta < \pi/2$.

(22) Find the period of :

(i) $\tan 5x$, (ii) $\sin (x/5)^\circ$, (iii) $\sin 2x + \cos x$.

(23) Construct (i) a sine function whose period is 3 and

(ii) a cotangent function whose period is 2.

(24) Examine the following functions for periodicity and if they are periodic, determine their periods.

(i) $\cos kx$, (ii) $\cos 7x$, (iii) $\tan x/2$, (iv) $\cos (x^2)$

(25) Draw the graphs of the following functions :

(i) $y = \sin 3x$, (ii) $y = \tan 2x$, (iii) $y = \sec 2x$.

(26) Fill in the blanks in the following :

(i) $\tan \theta$ is always negative if θ lies in the .....or ..... quadrant

(ii) If $\sin A = 3/5$ and $\pi/2 < A < \pi$ then $\cos A =$ .....

(iii) $\tan^2 \pi/3 - \sec^2 \pi/6 =$ ..... ($\operatorname{cosec}^2 \pi/6 - \cot^2 \pi/3$)

(iv) If $(-3, 4)$ is a point on the terminal arm of an angle θ whose initial arm is OX, then $\cos \theta + \sin \theta =$ .....

(v) If $5 \sin \theta = 4$ and $\pi/2 < \theta < \pi$ then $\sec \theta =$ .....

(vi) If $\sqrt{2} \cos \theta = 1$ and $0 < \theta < \pi/2$, then

$$\cos\left(\frac{\pi}{4} + \theta\right) = \text{.....}, \quad \sin\left(\frac{5\pi}{4} + \theta\right) = \text{.....}$$

(vii) If in a triangle ABC, $\angle A = \pi/2$ then $\cos^2 B + \cos^2 C =$ .....

(viii) If $3 \sec \theta = 5$ then $\operatorname{cosec} \theta \cdot \cot \theta =$ .....

(ix) If $A = 30^\circ$, $B = 135^\circ$ and $C = 60^\circ$, then $\cos^2 A - \sin^2 B + \tan^2 C =$ .....

(x) If $3 \tan A = 4$ and $\pi < A < 3\pi/2$, then $3 \sin A - 7 \cos A =$ .....

(27) State whether the following statements are true or false :

(i) $\cos 110^\circ - \cos 2^c$ is positive

(ii) If α and β are acute angles and $\cos \alpha > \cos \beta$, then $\alpha > \beta$.

(iii) If $\pi/2 < \theta < \pi$, then $\sec \theta - \cos \theta > 0$.

(iv) If $2 \sin A + 3 \cos A = 0$ then A must be in the second quadrant.

ANSWERS

(1) (i) 0, (ii) -1, (iii) 1, (iv) -1.

(2) $\sin\alpha = -4/5$, $\tan\alpha = 4/3$.

(3) $\sin\alpha = \dfrac{-k}{\sqrt{1+k^2}}$, $\cos\alpha = \dfrac{-1}{\sqrt{1+k^2}}$.

(4) $-10/3$. (6) (i), (iv), (vi) and (vii) have $+ve$ signs; the rest have $-ve$ signs. In (vii) note that $\pi < 4 < 3\pi/2$.

(20) (i) False, since $\sin x$ cannot be less than -1.

(ii) False, since $\left(a - \dfrac{1}{a}\right)^2 \geqslant 0$, and $\therefore\ a^2 + 1/a^2 \geqslant 2$.

We have $\cos^2 x = a^2 + 1/a^2 + 2 > 4$, but $\cos^2 x \leqslant 1$.

(iii) True. $(\sin x - 1)(\sin x - 2) = 0$. $\therefore\ \sin x = 1$ or 2.
But $\sin x \neq 2$; however $\sin x = 1$ is possible and so the statement is true.

(iv) False. $(\cos x - 2)(\cos x - 3) = 0$. $\therefore\ \cos x = 2$ or 3.
But $\cos x$ cannot be greater than 1.

(21) (i) True only for $x = y$, (ii) True only for $x = y$,
(iii) Impossible, (iv) True only if $x = 1$ or $x = -1$,
(v) False as $\sin\theta \leqslant 1$, $\cos\theta \leqslant 1$,
(vi) False as $\tan\theta < 1$ for $0 < \theta < \pi/4$,
(vii) True if $\pi/4 < \theta < \pi/2$.

(22) (i) $\dfrac{\pi}{5}$, (ii) 180°, (iii) 2π.

(23) (i) $\sin 2\pi x/3$, (ii) $\cot \pi x/2$.

(24) (i) Periodic with the period $2\pi/k$,
(ii) Periodic with the period $2\pi/7$,
(iii) Periodic with the period 2π, (iv) Not periodic.

(26) (i) 2nd, 4th, (ii) $-4/5$, (iii) $5/11$, (iv) $1/5$, (v) $-5/3$.

(vi) 0, -1, (vii) 1, (viii) $15/16$, (ix) $3\frac{1}{4}$, (x) $1\frac{4}{5}$.

(27) (i) True, (ii) False, (iii) False, (iv) False, A may lie in the 4th quadrant.

□ □ □

CHAPTER 3

Distance and Section Formulae

(1) Co-ordinates in a Plane :

(*i*) Rectangular Co-ordinates : Students have become familiar in article 1 of Chapter 2 with Rectangular Co-ordinates. We shall mainly use the system of Rectangular Co-ordinates in the present book. There is also, however, another system of Co-ordinates, known as Polar Co-ordinates, which will now be introduced.

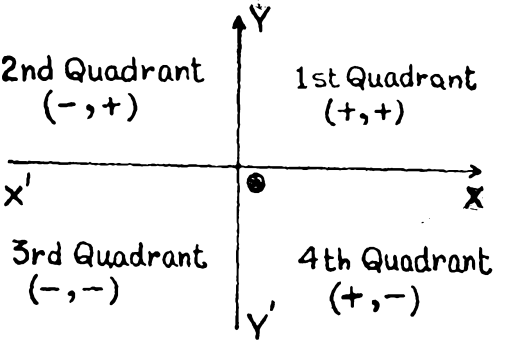


Fig. 3·1

(ii) Polar Co-ordinates : In this system a line OX is taken and a point O is chosen on it. If P be any point in the plane so that $l(OP) = r$ and $m(\angle XOP) = \theta$, then (r, θ) are called **the Polar Co-ordinates** of P. The point O is called the **pole.** The line OX is called the **initial** line. The distance OP is called the **radius vector** of P and the $\angle XOP$, measured in the anti-clockwise direction, is called the **vectorial angle** of P.

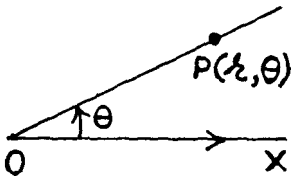


Fig. 3·2

(2) Relations between Rectangular Cartesian and Polar Co-ordinates :

From the $\Delta\ OMP$, we see that
$x = r\cos\theta, y = r\sin\theta$.

These relations hold for any value of θ lying in the interval $(0, 2\pi)$ provided r is taken as positive. Conversely, we can also find polar co-ordinates (r, θ) if the rectangular cartesian co-ordinates (x, y) of any point are known.

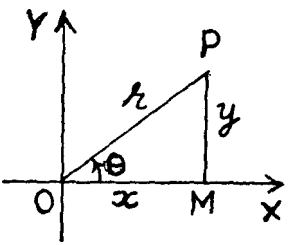


Fig. 3·3

For, $x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$, and $\tan\theta = y/x$.

$\therefore\ r = \sqrt{x^2 + y^2}$ and θ is given by $x = r\cos\theta$, $y = r\sin\theta$.

These relations enable us to change the co-ordinates of a point from one system to the other.

It can be seen that the co-ordinates of any particular point in the plane are not absolute. The co-ordinates of the point would change with a change in co-ordinate axes. Many times, the selection of a proper frame of axes is necessary to simplify calculations in Co-ordinate Geometry.

Ex. Find the polar co-ordinates of points whose rectangular cartesian co-ordinates are :—

(i) $(\sqrt{3}, 1)$, (ii) $(-\sqrt{3}, 1)$, (iii) $(-\sqrt{3}, -1)$, (iv) $(\sqrt{3}, -1)$.

Ans. (i) $(2, 30°)$, (ii) $(2, 150°)$, (iii) $(2, 210°)$, (iv) $(2, 330°)$.

(3) Interpretation of position vector as an ordered pair :

You have seen that the co-ordinates (x, y) of a point P enable us to fix the position of that point with reference to the axes chosen. It is also fixed if we know its distance $r\ (= OP)$ from the origin O as well as its direction, fixed by the angle θ. The directed segment OP is called **the position vector** of P and in the rectangular co-ordinate system where $P = (x, y)$, x being written first and y next, the position vector of P is taken as the ordered pair (x, y). We note that the ordered pairs (x, y) and (y, x) are the position vectors of two different points.

Note. We have frequently to deal with directed segments in further discussions. Hence it will be often convenient to use AB, PQ,....etc. to denote the lengths of the line segments AB, PQ,....etc. with their proper signs and this will be clear from the context in any discussion. For example, the cartesian co-ordinates of points are the lengths of line segments with proper signs in accordance with the usual conventions. Similarly the statement $AB \perp PQ$ often merely implies that seg $AB \perp$ seg PQ without any reference to their lengths or their directions including sense.

(4) Distance Formula : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points. We want to find the distance PQ in terms of x_1, y_1 and x_2, y_2. The result may first be proved by taking P and Q in the first quadrant. We will, however, consider the case when P lies in the third quadrant and Q lies in the first quadrant.

Let seg PM and seg QN be perpendicular to the x-axis. Let seg PL be perpendicular to seg NQ (produced in either direction if necessary).

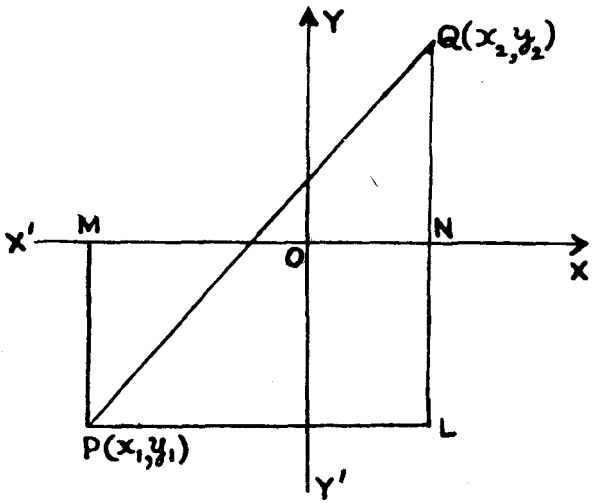


Fig. 3·4

Now in $\triangle PLQ$, $m \angle L = 90°$. Hence by Pythagoras Theorem we have $PQ^2 = PL^2 + LQ^2$. ..(i)

Now $PL = MN = MO + ON = ON + MO$;

i. e. $PL = ON - OM = x_2 - x_1$; ..(ii)

and $LQ = LN + NQ = NQ + LN$;

i. e. $LQ = NQ - NL = y_2 - y_1$. ..(iii)

Substituting these values of PL and LQ from (ii) and (iii) in (i), we get

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Here the distance PQ is considered as positive and the above formula gives the (numerical) distance between the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. The result can be seen to be true irrespective of the quadrants in which the points P and Q lie. Students may verify the result by taking P and Q to lie in different quadrants. In Fig. 3·4, students should note how OM (and not MO) is taken as x_1; similarly MP (not PM) and NL (not LN) are taken as y_1. The y-co-ordinate of a point is the distance from the foot of the perpendicular on the x-axis

to the point as measured in this direction and similarly for the x-co-ordinate.

Cor. 1. The distance of any point (x_1, y_1) from the origin is $\sqrt{x_1^2 + y_1^2}$.

Cor. 2. If $C \equiv (h, k)$, is a fixed point and $P(x, y)$ is a variable point such that $CP = r$, a constant, then a set of points P, where $(x - h)^2 + (y - k)^2 = r^2$, is the circle with C as its centre and r as its radius and we say that the equation of the circle (C, r) is $(x - h)^2 + (y - k)^2 = r^2$.

In particular, the equation of a circle with its centre at the origin and with radius r is $x^2 + y^2 = r^2$.

Illustrative Examples.

Ex. 1. Find the distance between the points $P(4, -7)$ and $Q(-1, 5)$

$PQ^2 = [4-(-1)]^2 + [-7-5]^2$

$= 25 + 144 = 169.$ $\therefore$ $PQ = 13.$

Ex. 2. Prove that the points $A(2, -2)$, $B(8, 4)$, $C(5, 7)$ and $D(-1, 1)$ form a rectangle.

We have $AB = \sqrt{(2-8)^2 + (-2-4)^2} = \sqrt{72} = 6\sqrt{2}.$

$BC = \sqrt{(8-5)^2 + (4-7)^2} = \sqrt{18} = 3\sqrt{2}.$

$CD = \sqrt{[5-(-1)]^2 + (7-1)^2} = \sqrt{72} = 6\sqrt{2}.$

$DA = \sqrt{[-1-2]^2 + [1-(-2)]^2} = \sqrt{18} = 3\sqrt{2}.$

$AB = CD$ and $BC = DA.$

$ABCD$ is a parallelogram.

Again $AC = \sqrt{(2-5)^2 + (-2-7)^2} = \sqrt{90}.$

Also $AB^2 + BC^2 = 72 + 18 = 90.$

$\therefore$ $AC^2 = AB^2 + BC^2.$

$\therefore$ $m(\angle ABC) = 90°.$ $\therefore$ $ABCD$ is a rectangle.

One may prove the result by merely considering the slopes of the lines, after getting introduced to the slopes of lines.

Ex. 3. Find the co-ordinates of the circumcentre of the triangle whose vertices are $(7, -1)$, $(5, 1)$ and $(-3, -7)$.

The circumcentre of a triangle is the centre of the circle passing through the vertices of the triangle. The circumcentre, therefore, is equidistant from the three vertices of the triangle.

Let $A(7, -1)$, $B(5, 1)$ and $C(-3, -7)$ be the three vertices of the $\triangle ABC$ with its circumcentre, at, say, $P \equiv (x_1, y_1)$.

We have

$$PA = PB = PC.$$

$$\therefore \quad PA^2 = PB^2 = PC^2. \qquad \ldots \text{ (i)}$$

By the distance formula, we have

$PA^2 = (x_1 - 7)^2 + (y_1 + 1)^2 = x_1^2 + y_1^2 - 14x_1 + 2y_1 + 50.$

$PB^2 = (x_1 - 5)^2 + (y_1 - 1)^2 = x_1^2 + y_1^2 - 10x_1 - 2y_1 + 26.$

$PC^2 = (x_1 + 3)^2 + (y_1 + 7)^2 = x_1^2 + y_1^2 + 6x_1 + 14y_1 + 58.$

With the help of the relations (i), we form any two convenient equations involving (x_1, y_1) and the solution of these simultaneous equations gives the required co-ordinates of the circumcentre.

Thus, $PA^2 = PB^2$ gives us the equation

$-14x_1 + 2y_1 + 50 = -10x_1 - 2y_1 + 26,$

i.e. $4x_1 - 4y_1 - 24 = 0$, i.e. $x_1 - y_1 - 6 = 0.$ ... (ii)

Similarly, $PA^2 = PC^2$ gives us the equation

$-14x_1 + 2y_1 + 50 = 6x_1 + 14y_1 + 58,$

i.e. $20x_1 + 12y_1 + 8 = 0.$ i.e. $5x_1 + 3y_1 + 2 = 0.$ ... (iii)

Solving equations (ii) and (iii) simultaneously, we have $x_1 = 2$ and $y_1 = -4$.

$\therefore$ the co-ordinates of the circumcentre are $(2, -4)$.

EXERCISE 3 (*a*)

(1) Show that the points $A(3, 8)$, $B(-11, 3)$, $C(-8, -2)$ are the vertices of an isosceles triangle.

(2) Find the circumcentre of the triangle whose vertices are
(i) $(4, 3)$, $(-2, 3)$, $(6, -1)$, (ii) $(2, 4)$, $(6, 10)$, $(12, 6)$.

(3) Prove that the triangle with vertices at the points $(0, 3)$, $(-2, 1)$ and $(-1, 4)$ is right-angled.

(4) Prove that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the vertices of a rectangle.

(5) Prove analytically, that the diagonals of a rectangle are equal.

[**Hint.** Let $ABCD$ be any rectangle. Take A as origin and AB, AD as the x and y axes. Choose the co-ordinates of B, C, D, as $(a, 0)$, (a, b) and $(0, b)$ respectively.]

(6) Prove that the points $P(-2, -1)$, $Q(1, 0)$, $R(4, 3)$ and $S(1, 2)$ are the vertices of a parallelogram.

(7) Prove, analytically, that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals.

[**Hint.** Choose proper axes and co-ordinates for the vertices as given in Fig. 3·5]

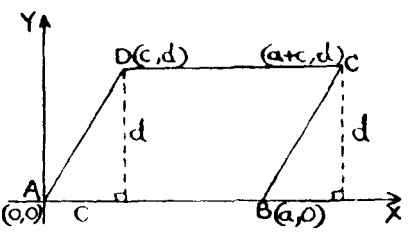


Fig. 3·5

(8) Show that the four points $P(1, -4)$, $Q(1, 0)$, $R(3, -2)$ and $S(-1, -2)$ form the vertices of a square and calculate the length of its diagonal.

(9) AOB is a right-angled triangle, right-angled at O. If C is the mid-point of seg AB, prove that $OC = CA = CB$.

[**Hint.** Take O as origin, OA as x-axis, OB as y-axis. Take $A = (a, 0)$ and $B = (0, b)$].

(10) D is the mid-point of the side BC of $\triangle ABC$.

Prove that $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

[**Hint.** Take D as origin, BC as the axis of x. Take $BC = 2a$, so that $B(-a, 0)$, and $C = (a, 0)$. Take $A = (h, k)$].

ANSWERS

(2) (i) $(1, -1)$, (ii) $(7, 5)$, (8) 4.

(5) Ratio of division of a line segment. If A and B are two given points and P is a point on the line joining them, the segment AB is said to be divided at P in the ratio AP / PB. The ratio is always to be written by dividing the length of the line segment from the first point to the point of division by the length of the line segment from the point of division to the second point. It is essential in this definition of ratio to regard distances measured in one direction, say AP as positive and in the opposite direction as negative. Fig. 3·6 illustrates the three possible cases which may arise.

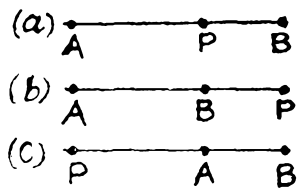


Fig. 3·6

In (a), seg AB is divided internally at P and in (b) and (c), it is divided externally at P. In (a), both AP and PB are measured in the same direction and hence the ratio of internal division is always positive. In (b) and (c), AP and PB are measured in opposite directions and hence the ratios of external division are always negative. It may be observed that in (b), the ratio is numerically greater than one and P lies on AB produced. In (c), the ratio is numerically less than one and P lies on BA produced. If the lengths of AP and PB are represented by m_1 and m_2 then the ratio of division is m_1/m_2 in (a) and $m_1/-m_2$ in (b) and (c).

(6) Section Formula. To find the co-ordinates of the point which divides the line segment joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ in a given numerical ratio m_1/m_2.

(I) Internal Division.

Let $P(x, y)$ divide seg AB where $A \equiv (x_1, y_1)$ and $B \equiv (x_2\ y_2)$, internally in the ratio $\frac{AP}{PB} = \frac{m_1}{m_2}$.

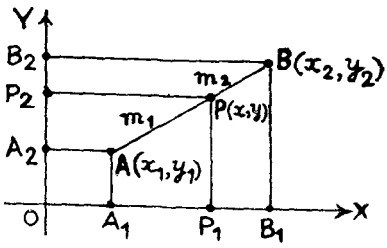


Fig. 3·7

Draw seg AA_1, seg PP_1, seg $BB_1 \perp OX$.

Now AA_1, PP_1, BB_1 are parallel and $APB, A_1P_1B_1$ intersect them.

$$\therefore \frac{AP}{PB} = \frac{A_1P_1}{P_1B_1}. \quad \therefore \frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}.$$

$$\therefore m_1 (x_2 - x) = m_2 (x - x_1).$$

$$\therefore m_1x_2 + m_2x_1 = (m_1 + m_2)x.$$

$$\therefore x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}\ ;\ \text{similarly } y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}.$$

Thus, the co-ordinates of $P(x, y)$ dividing the join of (x_1, y_1) and (x_2, y_2) internally in the ratio m_1/m_2 are

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \right).$$

(II) External Division.

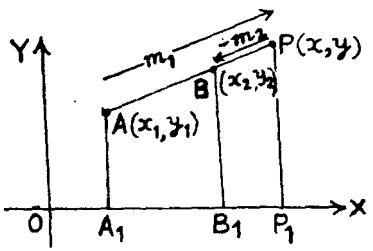


Fig. 3·8

If the point P divides AB, externally in the numerical ratio m_1/m_2 then $\frac{AP}{PB} = \frac{m_1}{-m_2}$; or $\frac{AP}{BP} = \frac{m_1}{m_2}$ both in magnitude and sign.

$$\therefore \frac{A_1P_1}{B_1P_1} = \frac{m_1}{m_2}. \quad \therefore \frac{x - x_1}{x - x_2} = \frac{m_1}{m_2}.$$

$$\therefore m_1 (x - x_2) = m_2 (x - x_1).$$

$$\therefore x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}\ ;\ \text{similarly } y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}.$$

Thus, the co-ordinates of $P(x, y)$ dividing the join of (x_1, y_1) and (x_2, y_2) externally in the numerical ratio $m_1 : m_2$ are

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \quad \frac{m_1y_2 - m_2y_1}{m_1 - m_2} \right).$$

Here we have taken the numerical ratio > 1.

Even if we take the numerical ratio < 1 and P to lie on BA produced, the formulae can be seen to be true.

Note. In this section formula it should be noted that m_1 which corresponds to the segment AP, is multiplied by the co-ordinates of B and m_2 which corresponds to the segment PB, is multiplied by the co-ordinates of A. We represent this in the Fig. 3·9 as an aid to memory.

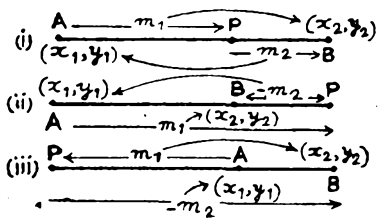


Fig. 3·9

Cor. 1. The mid-point of the line segment joining the two points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

[**Hint.** Take $\frac{m_1}{m_2} = 1$, i. e. $m_1 = m_2$.]

Cor. 2. The centroid G of $\triangle ABC$ whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is the point.

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

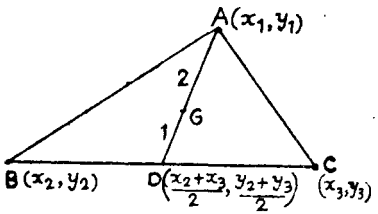


Fig. 3·10

[**Hint.** The centroid is the point of intersection of the three medians and it divides each median in the ratio 2 : 1, the longer segment being nearer to the vertex.]

Cor. 3. The co-ordinates of the points P and Q which divide the join AB of $A(x_1, y_1)$ and $B(x_2, y_2)$, harmonically in the same numerical ratio $\lambda : 1$ are

$$\left[\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda} \right], \left[\frac{x_1 - \lambda x_2}{1 - \lambda}, \frac{y_1 - \lambda y_2}{1 - \lambda} \right]$$

respectively.

Note. P and Q are said to divide seg AB harmonically at P and Q; also P and Q are called the harmonic conjugates of each other with respect to A and B.

Illustrative Examples.

Ex. 1. Find the co-ordinates of the point, which divides the segment joining the points $A(-3, 4)$ and $B(5, 6)$ internally in the ratio 3 : 2.

Let P be the required point.

Let x_P and y_P denote the abscissa and ordinate of P.

By using the formulae, we get

$$x_P = \frac{3(5) + 2(-3)}{3+2} = 9/5; \text{ and } y_P = \frac{3(6) + 2(4)}{3+2} = 26/5.$$

The co-ordinates of the required point are $(9/5, 26/5)$.

Ex. 2. Find the co-ordinates of the point which divides the segment joining the points $A(-1, -5)$ and $B(1, -2)$ externally in the ratio 4 : 3.

Let P divide seg AB externally in the ratio 4 : 3. Since the ratio is greater than one, B lies between A and P and $AP = 4$ and $PB = 3$.

$\therefore$ By using the formulae, we get

$$x_P = \frac{4(1)-3(-1)}{4-3} \text{ and } y_P = \frac{4(-2)-3(-5)}{4-3}$$

$$\therefore x_P = \frac{7}{1} \text{ and } y_P = \frac{7}{1}.$$

$\therefore$ the co-ordinates of the required point are $(7, 7)$.

Ex. 3. Find the co-ordinates of the point which divides externally the line segment joining the point $A(-1, 2)$ to $B(4, -5)$ in the ratio 1 : 3.

Let P divide seg AB externally in the ratio 1 : 3. Since the ratio is less than one, P lies on BA produced and we can take $AP = 1$ and $PB = 3$, which is -3 algebraically.

$\therefore$ By using the formulae, we get

$$x_P = \frac{1(4) - 3(-1)}{1-3} \text{ and } y_P = \frac{1(-5) - 3(2)}{1-3}.$$

$$x_P = -7/2 \text{ and } y_P = 11/2.$$

$\therefore$ the co-ordinates of the required point are $(-7/2, 11/2)$.

Ex. 4. If one end of a diameter of a circle with centre $(-3, 4)$ is $(5, -2)$, find the co-ordinates of the other end of the diameter.

Let seg AB be a diameter of the circle with C as its centre. We observe that C is the mid-point of seg AB.

Let $A \equiv (5, -2)$ and $C \equiv (-3, 4)$.

Let the co-ordinates of B be (x_1, y_1).

We have, by the mid-point formula

$$-3 = \frac{5 + x_1}{2} \text{ and } 4 = \frac{-2 + y_1}{2} \quad \therefore \quad x_1 = -11 \text{ and } y_1 = 10.$$

Hence the co-ordinates of the other end of the diameter are $(-11, 10)$.

Ex. 5. In what ratio does the y-axis divide the join of $(3, 5)$ and $(6, 7)$? Is the division internal or external ? Write down the co-ordinates of the point of division.

Let $A \equiv (3, 5)$ and $B \equiv (6, 7)$. Let the y-axis divide seg AB at P.

The point P divides seg AB in the ratio $AP/PB = \lambda/1$, say.

Let the co-ordinates of P be $(0, b)$.

We have by the section formula

$$0 = \frac{6\lambda + 1.3}{\lambda + 1} \quad \text{and} \quad b = \frac{7\lambda + 1.5}{\lambda + 1}.$$

From the former, we have $\lambda = -1/2$.

Substituting this value in the latter equation, we get

$$b = \frac{-7/2 + 1.5}{-1/2 + 1} \quad \text{i.e. } 3/1.$$

Since the ratio of division is negative, the division is external and the co-ordinates of the point of division are $(0, 3)$.

Ex. 6. If $(1, -2)$, $(5, 6)$, and $(-3, 2)$ are the mid-points of the sides of a triangle, find the co-ordinates of the vertices of the triangle.

Let $D(1, -2)$, $E(5, 6)$ and $F(-3, 2)$ be the given points. Through the points D, E, F, draw lines parallel to EF, FD and DE respectively so as to form the triangle ABC. We have to find the co-ordinates of A, B and C.

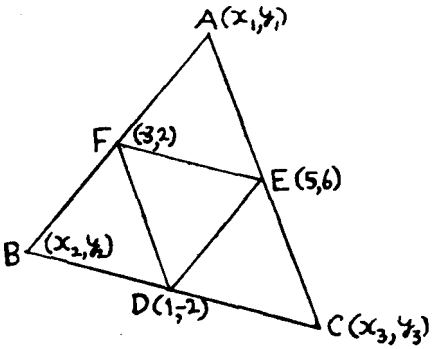


Fig. 3·11

Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$.

We observe that $DEAF$ is a parallelogram and hence seg DA and seg EF being diagonals have the same mid-point.

$$\therefore \quad \frac{x_1 + 1}{2} = \frac{-3 + 5}{2}; \quad \frac{y_1 - 2}{2} = \frac{2 + 6}{2}.$$

$\therefore \quad x_1 = 1, \; y_1 = 10. \qquad \therefore \quad A \equiv (1, 10).$

Similarly $BDEF$ is a parallelogram and hence seg BE and seg FD have the same mid-point.

$$\therefore \quad \frac{x_2 + 5}{2} = \frac{1 - 3}{2}; \quad \frac{y_2 + 6}{2} = \frac{-2 + 2}{2}.$$

$\therefore \quad x_2 = -7, \; y_2 = -6. \qquad \therefore \quad B \equiv (-7, -6).$

Similarly, $DCEF$ is a parallelogram and hence seg DE and seg CF have the same mid-point.

$$\therefore \quad \frac{-3 + x_3}{2} = \frac{1 + 5}{2}; \quad \frac{2 + y_3}{2} = \frac{-2 + 6}{2}.$$

$\therefore \quad x_3 = 9, \; y_3 = 2. \qquad \therefore \quad C \equiv (9, 2).$

$\therefore$ the co-ordinates of the vertices of the triangle, as required, are $(1, 10)$, $(-7, -6)$ and $(9, 2)$.

Ex. 7. Find the co-ordinates of the fourth vertex of the parallelogram whose three of the vertices are (− 3, 4), (0, − 4) and (5, 2).

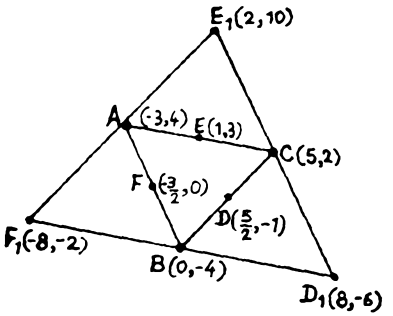


Fig. 3·12

Let $A \equiv (-3, 4)$, $B \equiv (0, -4)$, and $C \equiv (5, 2)$.

Let D, E, F be the mid-points of seg BC, seg CA and seg AB respectively.

Through the points A, B, C draw lines parallel to the opposite sides BC, CA and AB respectively.

Let ABD_1C, BCE_1A and CAF_1B be the three parallelograms, thus formed, having three of their vertices at A, B and C.

Let $D_1 \equiv (x_1, y_1)$, $E_1 \equiv (x_2, y_2)$ and $F_1 \equiv (x_3, y_3)$.

We observe that D is also the mid-point of seg AD_1.

Hence by the mid-point formula, we have

$$\frac{5}{2} = \frac{-3+x_1}{2}, \quad -1 = \frac{4+y_1}{2}.$$

$\therefore \quad x_1 = 8$ and $y_1 = -6$. $\qquad \therefore \quad D_1 \equiv (8, -6)$.

Similarly E and F are the mid-points of seg BE_1 and seg CF_1.

$$\therefore \quad 1 = \frac{0+x_2}{2}, \; 3 = \frac{-4+y_2}{2} \text{ and } \frac{-3}{2} = \frac{5+x_3}{2}, \; 0 = \frac{2+y_3}{2}.$$

$\therefore \quad x_2 = 2$, $y_2 = 10$ and $x_3 = -8$, $y_3 = -2$.

$\therefore \quad E_1 \equiv (2, 10)$ and $F_1 \equiv (-8, -2)$.

Hence there are three possible positions D_1 (8, − 6), E_1 (2, 10) and F_1 (− 8, − 2) for the fourth vertex of the parallelogram whose three vertices are (− 3, 4), (0, −4) and (5, 2).

EXERCISE : 3 (*b*)

(1) Find the co-ordinates of the point $P(x, y)$ which divides the line segment from

(i) A (1, 7) to B (6, − 3) internally in the ratio 2/3.

(ii) A (− 2, 1) to B (3, − 4) externally in the ratio 8/3.

(iii) A (−2, 3) to B (3, − 2) internally in the ratio 2/5.

(2) The join of A (1, 2) and B (2, 1) is harmonically divided at P and Q in the ratio 3 : 4. Find the co-ordinates of P and Q.

(3) Determine the two points of trisection P_1 (x_1, y_1) and P_2 (x_2, y_2) of the line segment joining A (3, −1) and B (9, 7).

(4) Find the co-ordinates of the centroid of the triangle whose angular points are (i) (1, − 2), (1, 4), (− 5, 3); (ii) (− 4, 6), (2, − 2), (2, 5).

(5) If $P(0, 4)$ divides the line segment joining $(-4, 10)$ and $(2, 1)$ internally, find the point which divides the line segment externally in the same ratio.

[**Hint** : Assume that P divides the join in the ratio $\lambda : 1$. Determine λ from the given data].

(6) In what ratio does the point $(-1, -1)$ divide the join of $(-5, -3)$ and $(5, 2)$? [**Hint** : Take the ratio as $\lambda : 1$].

(7) Find the ratio in which the join of $(-2, 1)$ and $(4, 4)$ is divided by the (i) x-axis, (ii) y-axis.

(8) In $\triangle ABC$, A is the point $(2, 5)$ and the centroid is at $(-1, 1)$. Find the co-ordinates of the mid-point of seg BC.

(9) Two vertices of a triangle are $(-4, 6)$, $(2, -2)$ and the centroid is $(0, 3)$. Find the third vertex.

(10) $(1, 3)$, $(2, a)$, $(b, 4)$ are the vertices of a triangle. If $(-3, 2)$ is its centroid, find a and b.

(11) C divides seg AB in the ratio $5 : 4$. If A is $(2, 7)$ and C is $(-3, -8)$, find the co-ordinates of B.

(12) A is $(-6, 10)$; B is a point such that the origin divides seg AB internally in the ratio $1 : 2$. Find the co-ordinates of B.

(13) Find the co-ordinates of G, the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3). If $(x_1, y_1) = (2, 3)$; $x_2 = 3$ and $y_3 = 2$ and G is $(0, 0)$, find y_2 and x_3.

(14) If A is $(1, 2)$ and B is $(7, -4)$ and P is $(a, 0)$ and A, P, B are collinear, find the ratio of the division of seg AB at P and show that $a = 3$.

(15) If the straight line through the points $A(4, 6)$ and $B(-7, -1)$ cuts the x-axis in the point C, find the ratio in which C divides seg AB. Further if D is the harmonic conjugate of C w. r. t. A and B, find the co-ordinates of D.

(16) If the points P and Q divide seg AB harmonically, prove analytically tha A and B divide seg PQ harmonically.

[**Hint** : Choose A as the origin and x-axis along AB.

Let $AB = a$ so that $B \equiv (a, 0)$. Let $\frac{AP}{PB} = \lambda$ so that $\frac{AQ}{QB} = -\lambda$, and Q divides seg AB in the ratio $-\lambda : 1$.

Show that $P \equiv \left[\frac{a\lambda}{\lambda + 1}, 0\right]$ and $Q \equiv \left[\frac{a\lambda}{\lambda - 1}, 0\right]$.

(17) If $(1, 3)$, $(-3, 5)$ and $(0, -1)$ are the mid-points of the sides of a triangle, find the co-ordinates of the vertices of the triangle.

(18) Show that $A \equiv (2, 2)$, $B \equiv (-3, 7)$, $C \equiv (5, -3)$ and $D \equiv (10, -8)$ are the vertices of a parallelogram.

[**Hint** : Show that the co-ordinates of the mid-points of the two diagonals are the same.]

(19) Find the possible co-ordinates of the fourth vertex of the parallelogram whose three of the vertices are at the points

(i) $(-5, 2)$, $(3, -1)$ and $(4, 3)$; (ii) $(1, 0)$, $(4, 3)$ and $(1, 2)$.

(20) Prove analytically, that the line segments joining the middle points of opposite sides of a quadrilateral and the line segment joining the middle points of its diagonals meet in a point and bisect one another.

[**Hint** : Take the vertices as (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4).

Show that $\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right)$

is a point of bisection in each case.]

(21) Fill in the blanks in the following :

(i) The cartesian co-ordinates of points A and B are respectively $(\sqrt{3}, -1)$ and $(0, \sqrt{5})$; hence their polar co-ordinates are ..... and ..... respectively.

(ii) The polar co-ordinates of C and D are respectively $\left(\frac{3}{2}, 60^\circ\right)$ and $(6, 210^\circ)$. Hence their cartesian co-ordinates are ..... and ..... respectively.

(iii) The co-ordinates of points on the y-axis at a distance of $\sqrt{41}$ from the point $(5, 8)$ are ..... and ..... .

(iv) If $A \equiv (4, 2)$, $B \equiv (6, -3)$ and $C \equiv (a, -4)$ are the vertices of a right-angled triangle, right-angled at A then $a =$ .....

(v) The co-ordinates of the circumcentre of a triangle whose vertices are $(-6, 0)$, $(0, 0)$ and $(0, 8)$ are (..... , ...) and its circum-radius is ..... .

(vi) If A is $(3, -2)$ and B is $(6, y)$ and $AB = 5$ units, then the possible values of y are ..... and ..... .

(vii) The co-ordinates of a point dividing the join of $(2, 3)$ and $(3, 2)$, in the ratio (a) 3 : 4 internally are (..... , ...), (b) 3 : 4 externally are (..... , ...).

(viii) Two of the vertices of a triangle are $(-2, 3)$, $(4, -5)$ and its centroid is $(2, 0)$. Hence the third vertex is (....., ...).

(ix) If one vertex of a triangle whose centroid is $(2, 4)$ is $(5, 8)$, then the mid-point of the side opposite to A is (....., ...).

(x) The join of A $(6, 3)$ and B $(-1, b)$ cuts x-axis at P $(3, 0)$ and the y-axis at Q. $\therefore$ $b =$ ... and $AQ : QB =$ ... : ...

(xi) The vertices of a parallelogram taken in order are $(3, 1)$, $(-1, 6)$, $(a, 5)$ and $(5, b)$. $\therefore$ $a =$ ... , $b =$ ..... and its diagonals intersect in the point (... , ...).

(xii) The internal bisector AD of a triangle ABC intersects BC in D. If $A \equiv (2, -2)$, $B \equiv (-1, 2)$ and $C \equiv (7, 10)$ then $BD : DC$ = ... : ... and $D \equiv$ (... , ...).

ANSWERS

(1) (i) (3, 3), (ii) (6, − 7), (iii) (− 4/7, 11/7).

(2) (10/7, 11/7), (− 2, 5). (3) $P_1 \equiv$ (5, 5/3), $P_2 \equiv$ (7, 13/3).

(4) (i) (− 1, 5/3), (ii) (0, 3). (5) (8, − 8). (6) 2/3.

(7) (i) Externally in the ratio 1 : 4. (ii) Internally in the ratio 1 : 2.

(8) (− 5/2, − 1). (9) (2, 5). (10) $a = -1$, $b = -12$.

(11) (− 7, − 20). (12) (12, − 20). (13) $y_3 = -5$, $x_8 = -5$.

(14) 1 : 2. (15) 6 : 1, (− 46/5, − 12/5). (17) (− 2, 9), (− 4, 1), (4, − 3).

(19) (i) (12, 0), (− 4, 6), (− 6, − 2); (ii) (− 2, − 1), (4, 1), (4, 5).

(21) (i) (2, 330°), ($\sqrt{5}$, 90°); (ii) (3/4, $3\sqrt{3}/4$), ($-3\sqrt{3}$, − 3);
(iii) (0, 4), (0, 12); (iv) − 11; (v) (− 3, 4), 5; (vi) 2, − 6;
(vii) (17/7, 18/7), (− 1, 6); (viii) (4, 2); (ix) (1/2, 2);
(x) − 4, 6 : 1; (xi) 1, 0 ; (2, 3). (xii) 5 : 13. (11/9, 38/9).

□ □ □

CHAPTER 4

Locus and Translation of Axes

(1) Geometrical Locus and its Algebraic Equation : We first define these two terms.

Locus : The co-ordinates (x, y) of any point in the plane are the perpendicular distances measured from the axes of reference Oy and Ox respectively. Thus, with every point in the plane we associate two real numbers x and y and they constitute an ordered pair of real numbers since we agree to write x first and then y. Thus, the ordered pairs of real numbers (x, y) and the points in a plane are in one-one correspondence. The locus, as students know in Geometry, is the path traced by the point when it moves under certain geometrical condition or conditions. Thus, *the locus is a set of points possessing a given property and hence is a subset of the set of all points* (x, y) *in the co-ordinate plane.*

Equation of a locus : *The equation of the locus of a point is the equation which is satisfied by the co-ordinates* (x, y) *of every point of the locus, and by the co-ordinates of no other point.*

It may be noted that every locus need not have an equation in (x, y) but every equation in (x, y) determines some locus.

A point (a, b) is said to satisfy an equation in x and y if when x and y are replaced by a and b, respectively, in the equation, the resulting members of the equation are equal. Thus a point (a, b) satisfies an equation $f(x, y) = 0$ if $f(a, b) = 0$ and conversely if $f(a, b) = 0$, then the point (a, b) belongs to the locus whose equation is $f(x, y) = 0$.

The co-ordinates (x, y) of any point of the locus are called **current co-ordinates.**

From the above definitions it follows that :

(i) if a point belongs to a locus, then its co-ordinates satisfy the equation of the locus; and

(ii) if the co-ordinates of a point satisfy the equation of a locus then the point belongs to the locus.

The two fundamental problems of Analytical Geometry are :

(i) Given a locus defined by some geometrical conditions to find its algebraic equation.

(ii) Given an equation, to find the corresponding locus.

We will consider the first problem here and attend to the second, whenever occasion arises.

We propose to illustrate the process of finding the equation of a locus by the following illustrations.

Illustrative Examples

Ex. 1. It is known in school Geometry that the locus of a point, which is such that its distances from two given points A, B are equal, is the perpendicular bisector of seg AB, say, CD. Now in order to obtain the equation of the locus CD, we take any point $P(x, y)$ and obtain an equation satisfied by the co-ordinates of $P(x, y)$ by using the geometrical condition $PA = PB$.

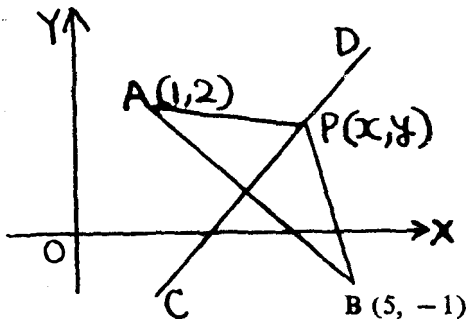


Fig 4.1

If $A \equiv (1, 2)$ and $B \equiv (5, -1)$, then $PA^2 = PB^2$ gives the equation

$$(x-1)^2 + (y-2)^2 = (x-5)^2 + (y+1)^2,$$

i.e. $$8x - 6y - 21 = 0. \quad \ldots \quad \ldots \quad (i)$$

Hence equation (i) represents the line CD and CD will be briefly called "the line $8x - 6y - 21 = 0$".

Thus, the subset of points P in a plane defined by $\{ P \mid PA = PB \}$ where A and B are given points in the plane is a locus which is a straight line bisecting seg AB at right angles.

The two concepts viz. the locus in Geometry and its equation in Algebra can be identified and we get the representation of the locus of points by an algebraic equation

Ex. 2 : It is known geometrically that the locus of a point in a plane, which is such that its distance from a given point is always the same is a circle. Now let us find its equation.

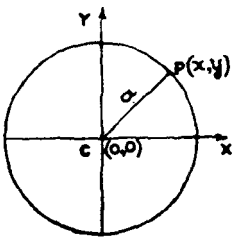


Fig. 4.2

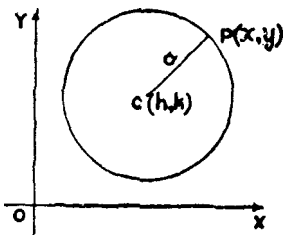


Fig. 4.3

Let the centre C of the circle be taken as origin and let the horizontal and vertical lines through the centre form the axes of reference. If $P(x, y)$ be any point on the circle, then the point P has the geometrical property, CP = constant, say, a;

$\therefore$ we have $CP^2 = a^2$.

$\therefore x^2 + y^2 = a^2$. ... ... (i)

$\therefore$ Equation (i) represents the circumference of a circle having its centre at the origin and radius = a.

Similarly the locus of a point, which is such that its distance from the given point (h, k) is equal to a, is the circumference of a circle and its equation is

$$(x - h)^2 + (y - k)^2 = a^2. \quad \ldots \quad \ldots \quad \text{(ii)}$$

Thus, the subset of points P in a plane defined by $\{ P \mid CP = a \}$ where C is a given point represents a locus which is the circumference of the circle with its centre at C and the radius equal to a.

The equation of the circle can be obtained if the co-ordinates of its centre and the length of its radius are known.

EXERCISE 4 (*a*)

(1) A set $\{ P(x, y) \mid y = 0 \}$ is a subset of all points, in the plane, whose ordinates are equal to zero and hence is the locus, known as the x-axis, whose equation is $y = 0$.

(2) A set $\{ P(x, y) \mid x = 0 \}$ is a subset of all points, in the plane, whose abscissae are equal to zero and hence is the locus, known as the y-axis, whose equation is $x = 0$.

(3) A set $\{ P(x, y) \mid x = h \}$ is a subset of all points in the plane, whose distances from the y-axis are equal to h, and hence is the locus which can be identified with a straight line parallel to the y-axis and at a distance h from it. The equation of this locus is $x = h$.

(4) A set $\{ P(x, y) \mid y = k \}$ is a subset of all points, in the plane, whose distances from the x-axis are equal to k, and hence is the locus which can be identified with a straight line parallel to the x-axis and at a distance k from it. The equation of this locus is $y = k$.

(5) A set $\{ P(x, y) \mid x = y \}$ is a subset of all points in the plane, whose distances from the axes are equal and hence is the locus which can be identified with the line bisecting the first and the third quadrants assuming that x and y are both of the same sign.

(6) A set of points P in a plane is defined by $\{ P \mid PA = PB \}$. If $A \equiv (3, 2)$ and $B \equiv (-2, 1)$, show that the equation of the locus is $5x + y - 4 = 0$.

(7) A set of points P in a plane is defined by $\{ P \mid CP = r \}$.

If $C \equiv (1, 2)$ and $r = 4$, show that the equation of the locus is

$$x^2 + y^2 - 2x - 4y - 11 = 0.$$

(8) Find the locus of P defined as follow :

$\{ P \mid$ distance of P from the x-axis $= 3 \}$. Ans. $y = 3$.

(2) Working rules for finding the equation of a locus.

The following main steps in finding the equation of a locus may be noted.

(i) Choose the origin and axes of reference if they are not given.

(ii) Take $P(x, y)$ to be any point of the locus.

(iii) Translate the given geometrical condition of the point P into an algebraic equation involving the co-ordinates (x, y) of the point P and the given constants specifying the locus.

Sometimes it is convenient to take the co-ordinates of the point P of the locus as $(\bar{x}, \bar{y})$ or (x', y') or (x_1, y_1), to avoid confusion with the current co-ordinates (x, y), which may occur in the given data. Ultimately the equation of the locus can be written by changing the co-ordinates of the point to (x, y).

We may add here a few simple examples illustrating the process of forming the equation to a locus.

Illustrative Examples

Ex. 1. Find the equation to the locus of a point which is such that its distance from the origin is equal to twice its distance from the axis of x.

Let $P(x, y)$ be any point of the locus.

The distance of P from the origin $= \sqrt{x^2+y^2}$.

The distance of P from the axis of x is equal to y.

$\therefore$ by the given condition in the problem, we have

$$\sqrt{x^2 + y^2} = 2y.$$

Squaring, the equation becomes $x^2 + y^2 = 4y^2$.

$\therefore$ the equation of the required locus is

$$x^2 - 3y^2 = 0, \text{ i. e. } x = \pm y\sqrt{3}.$$

Ex. 2. Find the equation to the locus of a point which is such that its distance from the axis of y is equal to its distance from the point $(1, 0)$.

Let $P(x, y)$ be any point of the locus.

The distance of P from the axis of y is equal to x.

The distance of P from $(1, 0) = \sqrt{(x-1)^2 + y^2}$.

The condition, which determines the locus, gives us

$$x = \sqrt{(x-1)^2 + y^2}, \quad \text{i. e. } x^2 = x^2 - 2x + 1 + y^2.$$

$\therefore$ the equation of the required locus is $y^2 - 2x + 1 = 0$.

Ex. 3. Find the locus of the point which is such that its distance from the point $(a, 0)$ is 'a' units more than its distance from the y-axis.

Let any point of the locus be $P(x, y)$.

The distance of P from $(a, 0) = \sqrt{(x-a)^2 + y^2}$.

The distance of P from the y-axis is equal to x.

The given condition gives the equation

$$\sqrt{(x-a)^2+y^2} = a + x.$$

Squaring and simplifying, we get

$x^2 - 2xa + a^2 + y^2 = a^2 + x^2 + 2xa.$

$\therefore$ the equation of the required locus is $y^2 = 4ax$.

This locus, geometrically, is known as a parabola.

Ex. 4. A is a point on the x-axis and B is a point on the y-axis such that $AB = 7$. Find the equation of the locus of a point P which divides seg AB externally in the ratio 2 : 3.

Let $P(x, y)$ be any point of the locus. The given condition in the problem is that

$$\frac{AP}{PB} = \frac{2}{-3}.$$

We have now to put this condition in terms of the co-ordinates (x, y) of the point P.

Let $A(a, 0)$ and $B(0, b)$ be the given points on the x and y-axes respectively.

$\therefore$ by the section formula, we have

$$x = \frac{2.0 - 3a}{2-3}, \quad y = \frac{2.b - 3.0}{2-3}.$$

$\therefore x = 3a, \quad y = -2b. \quad \therefore a = x/3$ and $b = -y/2$.

We are given that the distance $AB = 7$.

$\therefore \sqrt{(a-0)^2 + (0-b)^2} = 7. \quad \therefore a^2 + b^2 = 49.$

$\therefore (x/3)^2 + (-y/2)^2 = 49.$

$\therefore$ the equation of the locus of points P is

$$\frac{x^2}{9} + \frac{y^2}{4} = 49.$$

Ex. 5. $A(2, 5)$ and $B(4, -11)$ are two vertices of a $\triangle ABC$. Find the equation of the locus of the centroid of the triangle if the third vertex C is a point of the locus whose equation is $9x + 7y + 4 = 0$.

The given problem contains the current co-ordinates (x, y).

Hence, we assume the co-ordinates of the centroid, say, G to be (x_1, y_1).

The condition in the problem is that G is the centroid of the $\triangle ABC$ where $A \equiv (2, 5)$, $B \equiv (4, -11)$ and $C \equiv (x, y)$.

$$\therefore x_1 = \frac{2+4+x}{3} \text{ and } y_1 = \frac{5-11+y}{3}.$$

$\therefore 3x_1 = 6 + x$ and $3y_1 = -6 + y.$

$\therefore x = 3(x_1 - 2)$ and $y = 3(y_1 + 2).$

The equation satisfied by (x_1, y_1) can be obtained, since the equation satisfied by (x, y) is known.

We have $9x + 7y + 4 = 0.$

$\therefore 9 \times 3(x_1 - 2) + 7 \times 3(y_1 + 2) + 4 = 0.$

$\therefore 27x_1 + 21y_1 - 8 = 0.$

The locus of $G(x_1, y_1)$ is $27x + 21y - 8 = 0.$

Ex. 6. A is $(-2, 0)$ and Q is a point of the locus whose equation is $y^2 = 16x$. If P bisects seg AQ, find the equation of the locus of P.

The given problem contains the current co-ordinates (x, y). Therefore, we take the co-ordinates of the point P of the locus as (x_1, y_1).

The condition in the problem is that the point P is the midpoint of seg AQ where $A \equiv (-2, 0)$ and $Q \equiv (x, y)$.

$$\therefore \quad x_1 = \frac{-2 + x}{2} \quad \text{and} \quad y_1 = \frac{0 + y}{2},$$

$$\therefore \quad 2x_1 = -2 + x \quad \text{and} \quad 2y_1 = y.$$

$$\therefore \quad x = 2(x_1 + 1) \quad \text{and} \quad y = 2y_1.$$

The equation satisfied by (x_1, y_1) can be obtained since the equation satisfied by (x, y) is known. We have $y^2 = 16x$.

$$\therefore \quad 4y_1^2 = 32(x_1 + 1). \qquad \therefore \quad y_1^2 = 8(x_1 + 1).$$

$\therefore$ The locus of points P is $y^2 = 8(x + 1)$.

EXERCISE 4 (*b*)

(1) Find the equation of the locus of a point which is such that

(i) its distance from the origin = 3 units;

(ii) its distance from the point $(1, 2)$ = 4 units;

(iii) its distance from the y-axis = 2 units;

(iv) its distance from the x-axis = 5 units;

(v) its distance from the y-axis is half its distance from the origin.

(2) Interpret, geometrically, the following equations :

(i) $x = 1$, (ii) $x = -1$, (iii) $y = 2$,

(iv) $y = -2$, (v) $x = y$, (vi) $x = -y$, (vii) $x^2 + y^2 = 25$,

(viii) $(x - 1)^2 + (y - 2)^2 = 9$.

Draw the graph of the locus represented in each case.

(3) The equation of a certain locus is given by $x + 3y = k$.

If it is further given that the locus contains the point $(5, -6)$, determine the value of k.

(4) The equation of a certain locus is given by $2x + 3y = 6$. If the locus intersects the x-axis in A and the y-axis in B, find the co-ordinates of A and B.

(5) A and B are two given points whose co-ordinates are $(-3, 0)$ and $(3, 0)$. Find the equation of the locus of a point P which always satisfies the condition

$PA^2 + PB^2 = 50$. Interpret the equation geometrically.

(6) A and B are two given points. Find the equation of the locus of P if $PA^2 - PB^2 = 16$. Interpret the equation of the locus geometrically.

[**Hint.** Take O, the mid-point of seg AB as the origin; OB as the x-axis and OY, a line through O perpendicular to AB as the y-axis. Let the co-ordinates of P be (x, y) w. r. t. this set of axes. Take $AB = 2a$ so that $A \equiv (-a, 0)$ and $B \equiv (a, 0)$.]

(7) P is a point $(a\cos\alpha, a\sin\alpha)$ where α may have any value. Show that the equation of the locus of P is

$$x^2 + y^2 = a^2.$$

(8) A is a point on the x-axis and B is a point on the y-axis such that $OA + OB = 5$ where O is the origin of co-ordintates. Find the equation of the locus of points P which divide seg AB internally in the ratio 2 : 3.

(9) A is the point $(-1, 4)$ and Q is a point of the locus whose equation is $y^2 = 16x$. Find the equation of the locus of points P which divide seg AQ internally in the ratio 1 : 3.

(10) A is the point $(2, -1)$ and B is a point of the locus whose equation is $x^2 + y^2 = 16$. Find the equation of the locus of points P which divide seg AB externally in the ratio 3 : 2.

(11) $A(-5, -7)$ and $B(7, 1)$ are two vertices of a $\triangle ABC$. Find the equation of the locus of the centroid of the triangle if the third vertex C is a point of the locus whose equation is $2x - 3y = 21$.

(12) Obtain the equation of the locus of a point which divides the join of $(-1, -1)$ and a point on the circle with its centre at the origin and of radius 2 units, in the ratio 3 : 2.

(13) A straight line of length 2 units slides with its extremities on two perpendicular straight lines. Find the locus of its middle point.

[**Hint.** Let O be the point of intersection of the two perpendicular lines. Let the variable line intersect these lines in A and B. Take OA and OB as the x and y axes. If (x, y) are the co-ordinates of the mid-point of seg AB then $A \equiv (2x, 0)$ and $B \equiv (0, 2y)$. Use the condition that the length $AB = 2$].

(14) A point is such that its distance from the point $(1, 0)$ is 1 unit more than its distance from the y-axis. Show that the equation of its locus is $y^2 = 4x$.

(15) O is the origin and $A(x, y)$ is any point on the circle $x^2 + y^2 = 4$. P divides seg OA externally in the ratio 3 : 1. Find the equation of the locus of P and interpret it geometrically.

ANSWERS

(1) (i) $x^2 + y^2 = 9$, (ii) $(x-1)^2 + (y-2)^2 = 16$, (iii) $x = 2$, (iv) $y = 5$, (v) $3x^2 - y^2 = 0$.

(3) -13. (4) $(3, 0), (0, 2)$. (5) $x^2 + y^2 = 16$, a circle with centre $(0, 0)$ and radius equal to 4.

(6) $x = 4/a$, a constant. A straight line $\parallel$ to the y-axis.

(8) $2x + 3y = 6$. (9) $(y-3)^2 = 4x + 3$.

(10) $(x+4)^2 + (y-2)^2 = 144$. (11) $6x - 9y = 43$.

(12) $25(x^2 + y^2) + 20(x + y) - 28 = 0$. (13) $x^2 + y^2 = 1$.

(15) $x^2 + y^2 = 9$. A circle with centre at $(0, 0)$ and radius equal to 3.

(3) Translation of Axes : Formulae for Change of Origin :

It can be seen that the co-ordinates of a particular point in the plane are not absolute. The co-ordinates of the point would change with a change in co-ordinates axes. We, therefore, obtain here formulae for

the new co-ordinates of a point if the parallel axes through a point O' are taken.

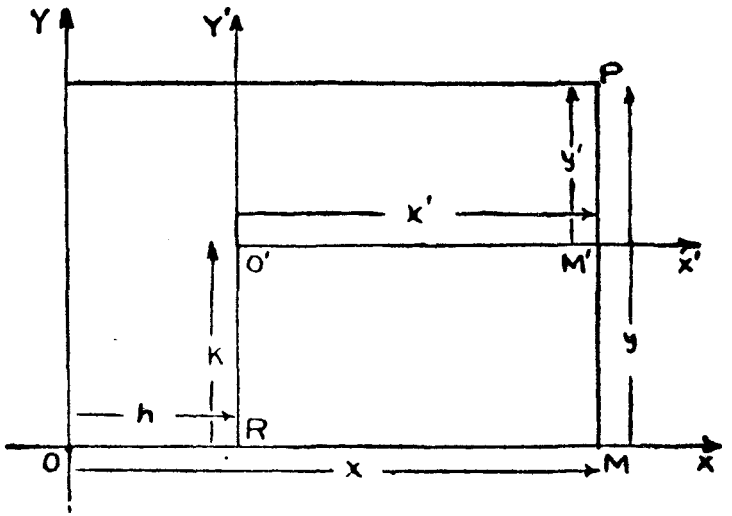


Fig. 4-4

Let O' have co-ordinates (h, k) w. r. t. a frame (OX, OY),

Let P have co-ordinates (x, y) ,, ,, ,,

Let P have co-ordinates (x', y') w. r. t. a new frame $(O'X', O'Y')$.

From the figure, we have

$$x = OM = OR + RM = OR + O'M' = h + x',$$

and $$y = MP = MM' + M'P = RO' + M'P = k + y'.$$

Thus, $x = h + x'$ and $y = k + y'$ give the relations between the co-ordinates of the same point P referred to the two systems and hence constitute the transformation formulae and would be found very useful when the parallel axes through another point are used.

The current co-ordinates of a point with reference to the new set of axes are often denoted by (X, Y). In this case, the transformation formulae become

$$x = h + X, \quad y = k + Y.$$

Many times it is possible to find the proper point O' (h, k) so that the new equation of a locus with reference to the parallel axes through O' assumes a simpler form.

We can then write the main points and lines with new current co-ordinates (X, Y) and by using the Transformation formulae

$$X = x - h, \quad Y = y - k,$$

we can get the main points and lines related to the locus with reference to the original set of axes.

Illustrative Examples

Ex. 1. If the origin is shifted to the point $(-3, 1)$, the axes remaining parallel to the original axes, find the co-ordinates of the point $(-3, 10)$ with reference to the new set of axes.

Let the current co-ordinates of a point referred to the new set of axes be (X, Y).

We have the transformation formulae

$$x = h + X, \; y = k + Y.$$

Here $x = -3 + X$, $y = 1 + Y$, since $(h, k) \equiv (-3, 1)$.

$\therefore \quad -3 = -3 + X$, $10 = 1 + Y$, since $(x, y) \equiv (-3, 10)$.

$\therefore \quad X = 0, \quad Y = 9.$

$\therefore$ the new co-ordinates of $(-3, 10)$ are $(0, 9)$.

Ex. 2. Find the co-ordinates of the point when its co-ordinates referred to the new parallel axes through $(3, -2)$ are $(6, 2)$.

With the usual notation, we have the transformation formulae

$$x = h + X, \; y = k + Y.$$

$\because \quad (h, k) \equiv (3, -2)$, and $(X, Y) \equiv (6, 2)$,

$\therefore \quad x = 3 + 6, \quad y = -2 + 2.$

$\therefore \; x = 9, \; y = 0.$

$\therefore$ the old co-ordinates of $(6, 2)$ are $(9, 0)$.

Ex. 3. If by shift of the origin, the co-ordinates of the point $(-7, 10)$ become $(2, 5)$, find the co-ordinates of the point where the origin is shifted.

With the usual notation, we have

$$x = h + X, \; y = k + Y.$$

$\therefore \quad -7 = h + 2$, $10 = k + 5$, since $(x, y) \equiv (-7, 10)$, and $(X, Y) \equiv (2, 5)$.

$\therefore \quad h = -9, \; k = 5.$

$\therefore$ the origin is shifted to the point $(-9, 5)$.

Ex. 4. What does the equation $x^2 + y^2 + 4x - 6y - 12 = 0$ become when the origin is shifted to $(-2, 3)$? Hence interpret the locus represented by the equation.

With the usual notation, we have the transformation formulae

$$x = h + X, \quad y = k + Y,$$

i. e. $\quad x = -2 + X, \quad y = 3 + Y$, since $(h, k) \equiv (-2, 3)$.

The transformed equation becomes

$$(-2 + X)^2 + (3 + Y)^2 + 4(-2 + X) - 6(3 + Y) - 12 = 0,$$

i. e. $\quad X^2 + Y^2 = 25.$

$\therefore$ the equation represents a circle with its centre at the new origin and radius $= 5$,

$\therefore$ the given equation represents a circle with its centre at $(-2, 3)$ and radius $= 5$.

EXERCISE 4 (e)

(1) If the origin is shifted to the point (3, 6), the axes remaining parallel to the original set of axes, find the co-ordinates of the points (i) (3, −5), (ii) (4, −2) with reference to the new set of axes.

(2) Find the co-ordinates of the points, when their co-ordinates referred to the new parallel axes through (5, 3) are (i) (−8, 2), (ii) (4, −3).

(3) When the origin is shifted, the point (1, 3) becomes the point (3, 4); find the co-ordinates of the new origin relative to the original set of axes.

(4) Transform the equation $x^2 + y^2 - 6x + 4y - 12 = 0$ when parallel axes through the point (3, −2) are taken. Interpret the equation.

(5) The equation of a locus is $xy + 2x - y - 2 = 0$. Under a translation of the axes in which the origin is transferred to (1, −2), find the new equation of the locus and show that the locus consists of two perpendicular lines.

(6) A point is such that its distance from $(2a, 0)$ is equal to its distance from the line $x + a = 0$. Find the equation of the locus. How will you translate the axes, so that the equation reduces to the form $Y^2 = kX$? Find the value of k.

(7) Find the equation of the locus of a point which is such that its distance from (3, 0) is equal to its distance from the line $x + 4 = 0$. Translate the axes to a new origin so that the equation reduces to the from $Y^2 = aX$. Find the value of a.

(8) By changing the origin to a suitable point O', show that the equation $y^2 - 4x + 6y + 17 = 0$ can be reduced to the form $Y^2 = 4aX$. Find a and the co-ordinates of O'.

(9) What does the equation of the curve $x^2 - 6x - 2y + 1 = 0$ become when the origin is changed to (3, −4)?

(10) Find the equation of the locus of a point such that its distance from the point (4, 3) bears a constant ratio, equal to 2, to its distance from the line $x + 5 = 0$. Also show that if the co-ordinate axes are translated and the new origin is properly chosen, the equation of the locus can be reduced to the form $3X^2 - Y^2 = 108$.

(11) Fill in the blanks in the following :—

(i) The locus of points bisecting radii of the circle $x^2 + y^2 = 9$ is the circle..... = ...

(ii) The equation of a locus is $y^2 = 27x$. The ordinates of points on the locus whose x-co-ordinates are (i) 3, (ii) 1/3 are.....and..... respectively.

(iii) The equation of a locus is $y^2 = 36x$. The abscissa of the point on the locus, whose ordinate is −9 is ..... .

(iv) The co-ordinates of points on the (i) x-axis (ii) y-axis which belong to the locus $x^2 + 4xy + y^2 + 6x - 6y + 8 = 0$ are (... , ...), (..., ...), and (... , ...), (... , ...) respectively.

(v) The equation of the locus of P such that $PA^2 - PB^2 = 10$ where $A \equiv (-6, 2)$ and $B \equiv (2, -6)$ is..... = 0.

(vi) A is $(3, -2)$ and B is $(2, 4)$. The locus of P such that $PA = 2PB$ is ..... $= 0$.

(vii) The equation of the locus given by $x^2 + y^2 - 6x + 10y - 48 = 0$, when the origin is shifted to $(3, -5)$ becomes.....

(viii) If the points $(-2, 1)$ and $(1, 7)$ belong to the locus $y = mx + c$ then $m =$ ....., $c =$ ..... .

ANSWERS

(1) (i) $(0, -11)$, (ii) $(1, -8)$. (2) (i) $(-3, 5)$, (ii) $(9, 0)$. (3) $(-2, -1)$.

(4) $X^2 + Y^2 = 25$. A circle with its centre at $(3, -2)$ and radius $= 5$.

(5) $XY = 0$, i. e. the axes $X = 0$ and $Y = 0$ which are at right angles.

(6) $y^2 = 6a(x - a/2)$. Translate the axes to the point $(a/2, 0)$, $k = 6a$.

(7) $y^2 = 14(x + 1/2)$. Translate the axes to the point $(-1/2, 0)$, $a = 14$.

(8) $O' \equiv (2, -3)$, $a = 1$. (9) $X^2 = 2Y$.

(10) $3(x + 8)^2 - (y - 3)^2 = 108$. Shift the origin to the point $(-8, 3)$.

(11) (i) $4x^2 + 4y^2 = 9$; (ii) -9, 9; -3, 3; (iii) 9/4, (iv) $(-4, 0)$, $(-2, 0)$; $(0, 2)$, $(0, 4)$; (v) $8x - 8y - 5 = 0$; (vi) $3x^2 + 3y^2 - 10x - 36y + 67$; (vii) $x^2 + y^2 - 82 = 0$; (viii) 2, 5.

□ □ □

CHAPTER 5

Straight Lines

(1) The Inclination of a Line : Definition : *The inclination (or the angle of inclination) of a line l (not parallel to the horizontal x-axis) is defined as the smallest positive angle formed by the rotation of the positive direction of the x-axis in a counter-clockwise direction till it first coincides with the line.*

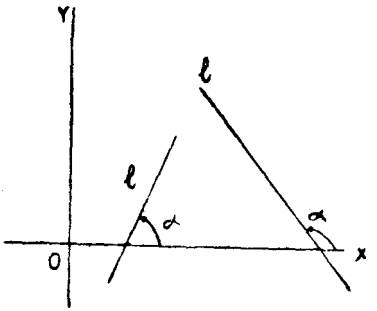


Fig. 5.1

If l is parallel to the x-axis, or coincides with the x-axis, its inclination is defined to be zero.

Note : If the measures of the angles of inclination of two lines are supplementary (instead of being the same), the two lines are often loosely referred to as equally inclined to the x-axis. Thus two lines having measures of inclination, say, 45° and 135° are referred to as equally inclined to the x-axis.

(2) The slope of a Non-vertical Line : Definition : *The slope (or gradient) of a non-vertical line is defined as the tangent of its inclination to the x-axis.*

We will usually use the letter m to denote the slope of a straight line. Thus we have $m = \tan \alpha$, if α is the measure of inclination of the line. The slope of a line is positive or negative according as the angle of inclination α is acute or obtuse. The slope of a line parallel to the x-axis (including the x-axis) will be naturally zero. The slope of a line parallel to the y-axis (including the y-axis) will not be finite, since $\tan \alpha$ is not finite when α equals 90°.

Formula for the Slope of a Non-vertical Line : You already know that the slope of a non-vertical line is the slope of any segment contained in the line. You also know that the slope of a non-vertical segment AB where $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is $(y_2 - y_1) / (x_2 - x_1)$, i. e. the ratio of the differences between the ordinates and the abscissae of A and B, the differences being taken in the same order. If the inclina-

tion of the line AB is α, then by the definition given in the above article, its slope is $\tan \alpha$.

We, therefore, have $\tan \alpha = \dfrac{y_2 - y_1}{x_2 - x_1}$. $x_1 \neq x_2$.

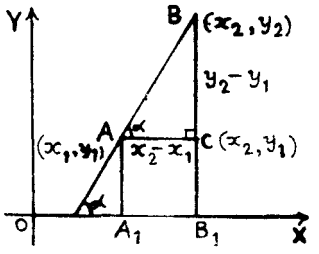


Fig 5·2 (i)

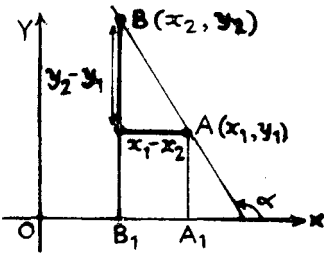


Fig. 5·2 (ii)

This relationship will be quite clear from the Fig. 5·2.

In Figures 5·2 (i), α is acute and its slope

$$\tan \alpha = \frac{y_2 - y_1}{x_2 - x_1} > 0.$$

In Figure 5·2 (ii). α is obtuse and its slope

$$\tan \alpha = \frac{y_2 - y_1}{x_2 - x_1} < 0.$$

(3) Slopes of Parallel and Perpendicular Lines

(i) If two lines are parallel, their inclinations are the same and hence they have the same slope. Thus two lines with slopes m and m' are parallel if $m = m'$ (Fig. 5·3).

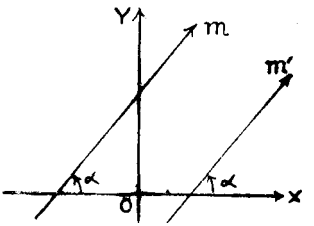


Fig. 5·3

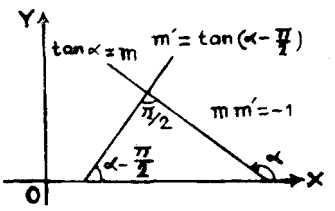


Fig. 5·4 (i)

(ii) If two lines are mutually perpendicular and if α is the inclination of one, then $\alpha \pm \pi/2$ is the inclination of the other. Hence if m and m' are the slopes of two mutually perpendicular lines then $m = \tan \alpha$ and

$m' = \tan(\alpha \pm \pi/2) = -\cot \alpha$.

$\therefore \quad m\, m' = -1.$

[In Fig. 5·4. (i), $\propto$ is obtuse and in Fig. 5·4 (ii), $\propto$ is acute].

Thus two lines are mutually perpendicular if the product of their slopes $= -1$.

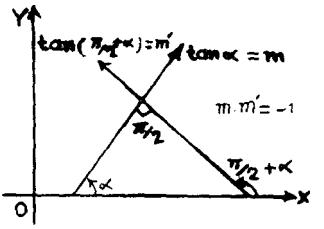


Fig. 5-4 (ii)

Illustrative Examples

Ex. 1. Find the slope and the angle of inclination of the line through the points (i) (2, 2), (−4, −4) and (ii) (−4, 3), (−4, 9).

(i) The slope $= \frac{-4-2}{-4-2} = \frac{-6}{-6} = 1.$

$\therefore$ if $\propto$ is the inclination, then $\tan \propto = 1$. $\therefore$ $\propto = \pi/4$.

(ii) The slope $= \frac{9-3}{-4-(-4)} = 6/0$, which is not finite.

$\therefore$ the line is parallel to the y-axis and hence is inclined at 90° to the x-axis.

Ex. 2. Show that the points (4, 5), (0, −3) and (−1, −5) are collinear.

Slope of the join of (4, 5) and (0, −3) is $\frac{-3-5}{0-4} = 2.$

,, ,, (4, 5) and (−1, −5) is $\frac{-5-5}{-1-4} = 2.$

,, ,, (0, −3) and (−1, −5) $= \frac{-5+3}{-1-0} = 2.$

$\therefore$ the slope of the join of any two of the three given points is the same. Hence the three points must be collinear.

Ex. 3. A (4, 3), B (−2, 5), C (2, −1), D (0, −7) are four given points. Show that $AB \perp CD$.

Slope m, of AB, $= \frac{5-3}{-2-4} = -1/3.$

Slope m', of CD, $= \frac{-7+1}{0-2} = 3.$

$\because$ $m\,m' = -1$, $\therefore$ $AB \perp CD$.

Ex. 4. If A (0, −2), B (3, 1), C (0, 4) and D (−3, 1) are given points show that $ABCD$ is a square.

m_1, the slope of AB, $= \frac{-2-1}{0-3} = 1.$

m_2, the slope of BC, $= \frac{1-4}{3-0} = -1.$

m_3, the slope of CD, $= \dfrac{4-1}{0+3} = 1$.

m_4, the slope of AD, $= \dfrac{-2-1}{0+3} = -1$.

$\therefore$ $m_1 = m_3$, $m_2 = m_4$ and $m_1 m_2 = -1$, $m_3 m_4 = -1$.

$\therefore$ $AB \parallel CD$, $BC \parallel AD$ and $AB \perp BC$, $CD \perp AD$.

It can also be shown that $AC \perp BD$.

$\therefore$ $ABCD$ is a square.

Ex. 5. Area of a triangle. If $A_1(x_1, y_1)$, $A_2(x_2, y_2)$ and $A_3(x_3, y_3)$ are the vertices of $\triangle A_1A_2A_3$, find the areas of (i) $\triangle OA_1A_2$, (ii) $\triangle OA_2A_3$, (iii) $\triangle OA_3A_1$ and (iv) $\triangle A_1A_2A_3$, where O is the origin.

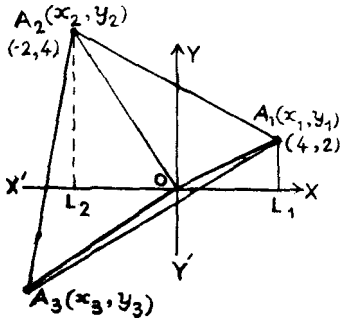


Fig. 5·5

We first illustrate the method by considering the case of $\triangle OA_1A_2$ where $A_1 = (4, 2)$, $A_2 = (-2, 4)$.

Let seg A_1L_1, seg $A_2L_2 \perp X'OX$.

Now area of $\triangle OA_1A_2$ = area of trap. $L_2L_1A_1A_2$ − area of $\triangle L_2OA_2$ − area of $\triangle OL_1A_1$,

[where in each case we go round the area in the same (here anti-clockwise) direction and the line segments are taken in the + ve directions of the axes],

$= \frac{1}{2}(L_2O + OL_1)(L_2A_2 + L_1A_1) - \frac{1}{2}L_2O.\ L_2A_2 - \frac{1}{2}OL_1.\ L_1A_1$ ... (i)

$= \frac{1}{2}[-(-2)+4]\ (4+2) - \frac{1}{2}.\ (+2)\cdot 4 - \frac{1}{2}\ 4\cdot(2)$

$= \frac{1}{2}.6.6 - 4 - 4 = 10.$

In general, where $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$, we have from step (i)

area of $\Delta OA_1A_2 = \frac{1}{2}\ [-(x_2) + x_1]\ (y_2 + y_1) - \frac{1}{2}\ (-x_2)\ y_2 - \frac{1}{2}\ x_1y_1$

$= \frac{1}{2}(-x_2 + x_1)\cdot(y_2 + y_1) + \frac{1}{2}\ x_2y_2 - \frac{1}{2}\ x_1\ y_1$

$= \frac{1}{2}(x_1y_2 - x_2y_1).$

Similarly, area of $\triangle OA_2A_1 = \frac{1}{2}(x_2y_1 - x_1y_2)$.

Thus the areas of triangles OA_1A_2 and OA_2A_1 are of opposite signs and are known as directed or oriented areas.

Convention about the sign of an area.

The sign of an area depends on the direction of rotation when we go round it. Thus in the above figure area of $\triangle OA_1A_2$ is + ve and area of $\triangle OA_2A_1$ is − ve since we go round them in the anti-clockwise and clockwise directions respectively.

One can see from the above diagram, where the point O lies within the angle formed by the rays A_2A_3 and A_2A_1, that the areas of triangles OA_1A_2 and OA_3A_3 will be of the same + ve sign since the rotation in the cases of both the triangles is anti-clockwise. One can again observe by drawing various diagrams that if O lies within the triangle $A_1A_2A_3$ (as in the above diagram) then the areas of triangles OA_1A_2,

OA_2A_3, OA_3A_1 and $A_1A_2A_3$ will all be of the same sign; whereas all of them will not be of the same sign if O lies outside the $\triangle A_1A_2A_3$. Evidently if O lies in a side of a triangle, then the corresponding area is zero. However, in every case we have the relation : area of $\triangle A_1A_2A_3$ = area of $\triangle OA_1A_2$ + area $\triangle OA_2A_3$ + area of $\triangle OA_3A_1$ provided each area is taken as an oriented area and hence with its proper sign.

Hence area of $\triangle A_1A_2A_3$ is given by

$\frac{1}{2}(x_1y_2 - x_2y_1) + \frac{1}{2}(x_2y_3 - x_3y_2) + \frac{1}{2}(x_3y_1 - x_1y_3)$,

i. e. by $\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$.

Note : The area of any polygon whose vertices are given can be similarly obtained by dividing the polygon into a number of triangles and adding their directed areas. The expression for the area of a polygon holds only if the vertices are taken in order; if, therefore, the order of the vertices is not known it must first be determined by first plotting the given points.

EXERCISE 5 (*a*)

(1) Find the slope and inclination of the line connecting the points :

(i) $(-1, \sqrt{3})$, $(2, -2\sqrt{3})$; (ii) $(5, 6)$, $(10, 6)$; (iii) $(6, 5)$, $(6, 10)$.

(2) $A(2, 3)$. $B(-3, -7)$, $C(5, -3)$ and $D(-6, -1)$ are given points. State which of the lines AB, AC, AD, BC, BD, CD are (i) parallel, (ii) perpendicular. Plot the points and give a particular name for the quadrilateral formed by these points.

(3) If $(7, a)$, $(-5, 2)$, $(3, 6)$ are collinear points, find a.

[**Hint** : The slopes of the joins of any two of the three points are the same.]

(4) Find a, if the triangle formed by $A(4, 3)$, $B(6, -2)$ and $C(a, -3)$ is right-angled at A.

[**Hint** : Slope of $AB \times$ slope of $AC = -1$].

(5) Write the condition that a line joining $P(x, y)$ to $A(1, -2)$ be parallel to the line through $C(5, 6)$ and $D(7, -3)$.

(6) Write the condition that the line through $P(x, y)$ and $A(5, 6)$ be perpendicular to the line through $C(4, 5)$ and $D(-2, -5)$.

(7) Prove, only by considering the slopes, that the diagonals of a square $ABCD$ are perpendicular to each other.

[**Hint** : Choose A as the origin and AB, AD as the x and y axes. Take the co-ordinates of B, C, D as $(a, 0)$, (a, a) and $(0, a)$ respectively.]

(8) Show that the points $A(7, 5)$, $B(2, 3)$, $C(6, -7)$ are the vertices of a right-angled triangle of 29 square units area.

(9) Show that the points $A(3, 0)$, $B(0, 2)$, $C(-6, 8)$ and $D(9, -2)$ are the vertices of a trapezium.

(10) A straight line joining $A(-4, 5)$ and $B(6, 10)$ is divided by the straight line joining $P(-1, 13)$ and $Q(1, 1)$ at a point R.

Show that $\frac{AR}{RB} = 2/3$ and $R \equiv (0, 7)$.

[**Hint :** Take $\frac{AR}{RB} = k/1$. Show that $R \equiv \left(\frac{6k-4}{k+1}, \frac{10k+5}{k+1}\right)$.

Determine k from the fact that the slopes of PR and PQ are the same].

(11) Find the areas of the triangles whose vertices are :
(i) (−8, −2), (−4, −6) and (−1, 5),
(ii) (0, 4), (−8, 0) and (−1, 4) (iii) (−7, 5), (1, 1) and (−3, 3).
Explain the geometrical significance of the answer of example (iii).

(12) Find the area of the quadrilateral whose vertices are
(−1, −1), (−4, −2), (−5, −4) and (−2, −3).

(13) Show that the area of the pentagon whose vertices are :
(1, 5), (−2, 4), (−3, −1), (2, −3) and (5, 1) is 40 units.

(14) Prove that the points (a, 0), (0, b) and (1, 1) are collinear if $1/a + 1/b = 1$.

(15) If $P(x, y)$ is any point lying on a line passing through the point $A(a, 0)$. and $B(0, b)$, show that $bx + ay - ab = 0$.

ANSWERS

(1) (i) $-\sqrt{3}$, 120°; (ii) 0, 0; (iii) not finite, 90°..

(2) $AD \parallel BC$, $BD \parallel CA$, $AD \perp AC$, $AD \perp BD$, $BD \perp BC$, $BC \perp AC$ $AB = CD$. $BCAD$ is a rectangle.

(3) $a = 8$. (4) $a = -11$. (5) $\frac{y+2}{x-1} = \frac{-9}{2}$. (6) $\frac{y-6}{x-5} = \frac{-3}{5}$.

(11) (i) 28, (ii) 2, (iii) 0, (12) 5.

(4) Equations of Straight Lines in some standard forms. We will now consider the simplest locus, namely, the straight line. The common property (or the condition) satisfied by all points on a straight line, in general, is that the slope of the line is the same as the slope of any segment joining any two points on it. By considering this property we will obtain some standard forms of the equations of straight lines.

(1) Equations of lines parallel to the co-ordinate axes :

(i) Let $P(x, y)$ be any point on a line parallel to the y-axis at a distance h from it For any position of the point P on this line, and for no other point, the x co-ordinate is always constant and is equal to h.

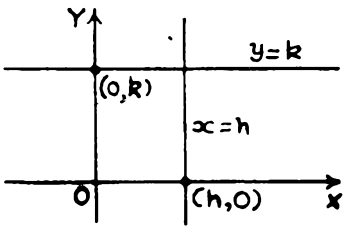


Fig. 5·6

$x = h$ is the equation of the line parallel to the y-axis and at a distance h from it.

(ii) Similarly $y = k$ is the equation of the line parallel to the x-axis and at a distance k from it.

Cor. $y = 0$ is the equation of the x-axis and $x = 0$ is the equation of the y-axis.

(2) Two-points form. The equation of the line passing through two points. Let the straight line pass through the two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$.

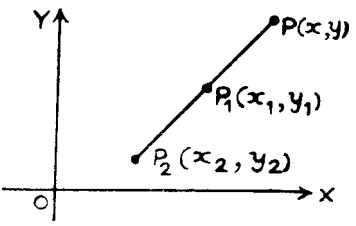


Fig. 5.7

$\therefore$ Slope of the line P_1P_2 is

$$\frac{y_1 - y_2}{x_1 - x_2}, \quad x_1 \neq x_2. \qquad \ldots \text{(i)}$$

Let $P\ (x, y)$ be any point on the line.

$$\text{Slope of } PP_1 = \frac{y - y_1}{x - x_1}. \qquad \ldots \qquad \ldots \text{(ii)}$$

$\because$ the slope of PP_1 = slope of P_1P_2,

$\therefore$ from (i) and (ii), we have $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}, \quad x_1 \neq x_2.$

[If $x_1 = x_2$, the line becomes $x = x_1$].

$\therefore \quad \dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$ or $\dfrac{x - x_1}{x_1 - x_2} = \dfrac{y - y_1}{y_1 - y_2}$

represents the equation of the line passing through the two points (x_1, y_1) and (x_2, y_2).

Example. Find the equation of the line passing through the points

(i) $(-2, 6)$ and $(3, 5)$, (ii) $(8, -2)$ and $(7, -2)$.

Ans. (i) $x + 5y - 28 = 0$. (ii) $y + 2 = 0$.

(3) Slope-point Form. The equation of a line having a slope m and passing through the given point $Q\ (x_1, y_1)$.

Let $P\ (x, y)$ be any variable point on the line. Q is (x_1, y_1).

$\therefore$ slope of $PQ = \dfrac{y - y_1}{x - x_1}$. $\qquad \ldots \text{(i)}$

But it is given that the slope of the line is m. $\qquad \ldots \text{(ii)}$

From equations (i) and (ii), we get $\dfrac{y - y_1}{x - x_1} = m$;

i. e. $$y - y_1 = m\ (x - x_1)$$

as the equation of the line passing through $Q\ (x_1, y_1)$ and having a slope m.

(4) Slope-intercept Form. The equation of a line of slope m and an intercept c on the y-axis.

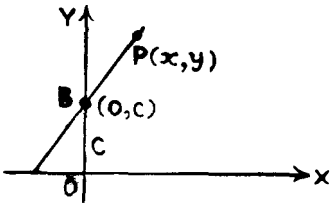


Fig. 5.8

Let a straight line of slope m intersect the y-axis in B. Let OB, the intercept on the y-axis, be c. Then the co-ordinates of B are $(0, c)$. Let P (x, y) be any variable point on the line.

$\therefore$ slope of $BP = \dfrac{y-c}{x-0}$. .. (i)

But the slope of the line is given to be m. .. (ii)

Equating (i) and (ii), we get the required equation of the line as

$$\frac{y-c}{x} = m \quad \text{or } y = mx + c.$$

Thus $y = mx + c$ is the equation of the line having a slope m and an intercept c on the y-axis.

Cor : If $c = 0$, the intercept $OB = 0$; and the straight line passes through the origin. Hence $y = mx$ is the equation of the line having a slope m and passing through the origin.

(5) Inclination-point Form. The equation of a line passing through a given point and having a given inclination.

Let a straight line passing through a given point $P_1(x_1, y_1)$ be inclined at an angle of measure θ to the x-axis. The slope of the straight line, therefore, is $\tan\theta$.

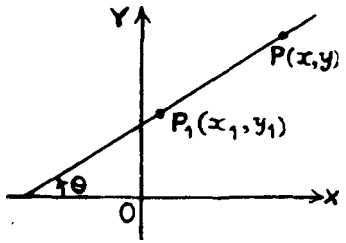


Fig. 5.9

.. (i)

If $P(x, y)$ is any point on this line, then slope of PP_1 is

$$\frac{y-y_1}{x-x_1} \qquad \ldots \qquad \ldots \text{(ii)}$$

$$\therefore \quad \frac{y-y_1}{x-x_1} = \tan\theta \text{ or } \frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}$$

is the equation of the line passing through the point (x_1, y_1) and inclined at an angle θ to the x-axis.

If in the Fig. 5·9 the point P (x, y) is taken at a distance $|r|$ from the point P_1 then the line takes the form

$$\frac{x - x_1}{r} = \cos\theta, \quad \frac{y - y_1}{r} = \sin\theta.$$

$$\therefore \quad \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

or $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ express the co-ordinates (x, y) of any point P on the line in terms of r, $|r|$ being the distance of the point from a given point on the line, and constitute the parametric equations of the line. In these equations x_1, y_1 and θ are known, and the current co-ordinates are expressed in terms of one unknown, called the parameter r. The position of the point on the line will vary corresponding to a change in the value of the parameter r. Note that $r > 0$ if $y > y_1$ since $r(y - y_1) = r^2\sin\theta > 0$ as $0 < \theta < \pi$. Obviously $r < 0$, if $y < y_1$.

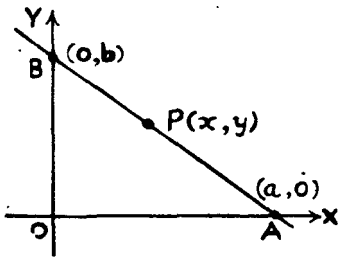


Fig. 5·10

(6) Two intercepts Form. The equation of the line having intercepts a and b on the co-ordinate axes.

Let the straight line intersect the co-ordinate axes in A and B respectively. OA and OB are then called the intercepts of the line on the axes of x and y respectively.

Let $OA = a$ and $OB = b$ so that

$A \equiv (a, 0)$ and $B \equiv (0, b)$.

$$\therefore \text{ the slope of the line } AB = \frac{b-0}{0-a} = \frac{-b}{a}. \qquad \ldots \text{ (i)}$$

Let $P(x, y)$ be any variable point on the line so that the slope of

$$AP = \frac{y-0}{x-a}, \text{ i.e. } \frac{y}{x-a}. \qquad \ldots \text{ (ii)}$$

$\because$ the slope of AP = the slope of AB, we get from (i) and (ii)

$$\frac{y}{x-a} = \frac{-b}{a}.$$

$$\therefore ay = -bx + ab. \qquad \therefore bx + ay = ab.$$

Dividing by ab, we get the equation of a line having intercepts, a, b on the co-ordinates axes as

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a, b, \neq 0.$$

Remarks. (i) If the x-intercept does not exist (i. e. is infinite) the line become: $y = b$. (ii) If the y-intercept is infinite then the line becomes $x = a$.

(7) Normal or Perpendicular Form. The equation of a line at a given perpendicular distance from the origin, when the inclination of this perpendicular is known.

Let the straight line be at a perpendicular distance p from the origin. Let the inclination of this perpendicular, say ON, be α. The co-ordinates of N, the foot of the perpendicular, in terms of the given constants, are $(p\cos\alpha, p\sin\alpha)$.

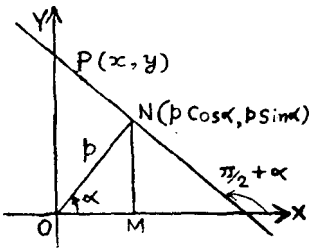


Fig. 5·11

The inclination of the line to the x-axis is $\left(\frac{\pi}{2} + \alpha.\right)$

$\therefore$ the slope of the line $= \tan\left(\frac{\pi}{2} + \alpha\right)$, i. e. $-\cot\alpha$. .. (i)

Let $P(x, y)$ be any point on the line.

Slope of NP is $\frac{p\sin\alpha - y}{p\cos\alpha - x}$. .. .. .. (ii)

From (i) and (ii), $\frac{p\sin\alpha - y}{p\cos\alpha - x} = -\frac{\cos\alpha}{\sin\alpha}$,

i. e. $x\cos\alpha + y\sin\alpha = p \quad (\sin^2\alpha + \cos^2\alpha)$.

$\therefore$ $x\cos\alpha + y\sin\alpha = p$ .. .. (iii)

represents the equation of the line at a perpendicular distance p from the origin, the perpendicular being inclined at an angle α to the x-axis.

Sign of the Perpendicular p. We shall now introduce a convention about the sign of the perpendicular from the origin to the line and the mode of measuring the angle of inclination of this perpendicular to the x-axis. The convention should be such as will ensure the right sign for the intercepts of the line on the co-ordinate axes. We adopt the following convention.

Convention. The form $x \cos \alpha + y \sin \alpha = p$ gives the correct sign for the x and y intercepts in all cases of lines provided α is measured from 0 to 2π and provided the perpendicular from the origin is always considered as positive.

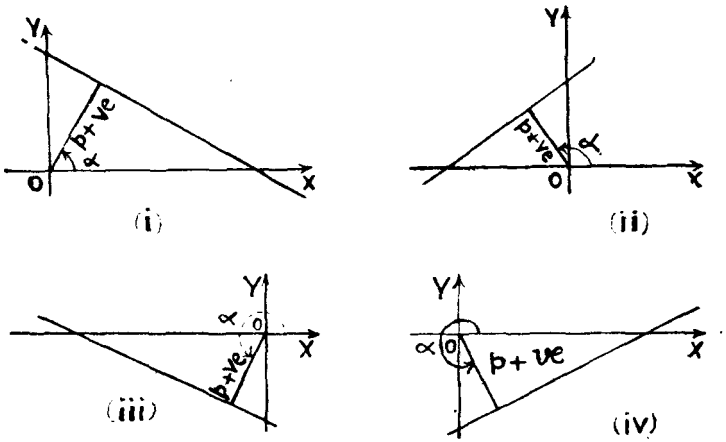


Fig. 5·12

In Fig. 5·12 (i), α is acute, $\therefore$ $\cos \alpha$, $\sin \alpha$ are + ve. $\therefore$ x-intercept $p/\cos \alpha$, and y-intercept, $p/\sin \alpha$, are + ve.

In Fig. 5·12 (ii), α is obtuse. $\therefore$ $\cos \alpha$ is − ve and $\sin \alpha$ is + ve. $\therefore$ x-intercept is − ve and y-intercept is + ve.

In Fig. 5·12 (iii), α lies in the third quadrant. $\therefore$ $\cos \alpha$ and $\sin \alpha$ are both − ve. $\therefore$ x and y-intercepts are both − ve. Similar results are true in the case of Fig. 5·12 (iv).

EXERCISE 5 (*b*)

Ex. 1 : Determine the equation of the line

(i) 5 units to the left of the y-axis,

(ii) 7 units above the x-axis,

(iii) passing through the origin and having an inclination = 60°,

(iv) passing through the origin and the point (2, 3),

(v) having y-intercept = 2 and slope = 1.

(vi) with slope 2/3 and passing through the point (−4, 5),

(vii) passing through the two points (3, −1) and (0, 6),

(viii) with intercepts 2 and −3 on x and y axes respectively,

(ix) for which the length of the perpendicular from the origin is $1/\sqrt{2}$, the inclination of this perpendicular being $\pi/4$,

(x) drawn through P (3, 4) and inclined to the x-axis at an angle $3\pi/4$. Find also the co-ordinates of the two points on it on opposite sides of P at a distance $\sqrt{2}$ from it.

Ans. (i) $x = -5$, (ii) $y = 7$, (iii) $y = x\sqrt{3}$, (iv) $3x - 2y = 0$, (v) $y = x + 2$, (vi) $2x - 3y + 23 = 0$, (vii) $7x + 3y - 18 = 0$, (viii) $3x - 2y = 6$, (ix) $x + y = 1$, (x) $x + y - 7 = 0$, (2, 5) and (4, 3).

Ex. 2 : Interpret, geometrically, the following equations :

(i) $x = 0$, (ii) $y = 0$, (iii) $x + y = 0$, (iv) $x - y = 0$, (v) $x = -2$, (vi) $y = 3$, (vii) $x/2 - y/3 = 1$, (viii) $y = x + 2$, (ix) $2x = 3y$, (x) $x \cos 30^\circ + y \sin 30^\circ = 3$.

Ans : (i) y-axis, (ii) x-axis, (iii) the line bisecting the second and fourth quadrants, (iv) the line bisecting the first and third quadrants, (v) a line parallel to the y- axis and at a distance of 2 units to its left, (vi) a line parallel to the x-axis and at a distance of 3 units above it, (vii) a line having intercepts 2 and -3 on the x and y axes respectively, (viii) a line with slope 1 and y-intercept = 2, (ix) a line through the origin and the point (3, 2), (x) a line at a perpendicular distance of 3 units from the origin, the inclination of this perpendicular being 30°.

(5) General form of the equation of a line. Proposition. The equation of any straight line is a first degree equation in the co-ordinates x, y.

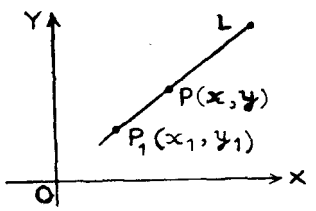


Fig. 5·13

Let P_1 (x_1, y_1) be some fixed point taken on a straight line L.

Let P (x, y) be any variable point on this line. For a position of the point P lying on this straight line and for no other point, the gradient or slope of PP_1 is always constant, say, m. Translating this geometrical condition under which the point P belongs to the line L, we have $\dfrac{y - y_1}{x - x_1} = m$.

$\therefore\ m\ (x - x_1) - (y - y_1) = 0.$

$\therefore$ the equation of a line, in general, is

$$mx - y + (y_1 - mx_1) = 0. \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

The equation (i), we have found, is of the form

$Ax + By + C = 0$ where A, B, C are constants and is thus a linear equation in x, y.

We have, thus, proved that the equation of any line is a first degree equation. We now prove the Converse viz,

Every equation of the first degree in x, y represents a straight line.

Let the general equation of the first degree viz.

$Ax + By + C = 0$ where A, B, C are constants, such that A and B are not simultaneously equal to zero, represent some locus.

Let $P_1\ (x_1, y_1)$ be some definite point taken on this locus.

$\therefore\ Ax_1 + By_1 + C = 0.$ .. .. (i)

Let $P\ (x, y)$ be any variable point on the locus.

$\therefore\ Ax + By + C = 0.$ .. .. .. (ii)

Subtracting (i) from (ii), we get $A\ (x - x_1) + B\ (y - y_1) = 0.$

$$\therefore\ \frac{y - y_1}{x - x_1} = -\frac{A}{B}\quad (B \neq 0).$$

[When $B = 0$, the locus is the line $x = x_1$].

Now the *R. H. S.* is constant and the *L. H. S.* is the slope of the join of P_1, some definite point on the locus, and P, a variable point on the locus.

$\therefore$ the point P can obviously lie only on a straight line.

$\therefore$ the locus represented by $Ax + By + C = 0$ can only be a straight line.

Obviously A and B cannot be zero simultaneously since the equation then ceases to be linear. If $A = 0$, the line is $y = -C/B$ and if $B = 0$, the line is $x = -C/A$.

Thus, we have proved that (i) a straight line is represented by a linear equation and (ii) a linear equation represents a straight line.

Remarks. The general equation of a line $Ax + By + C = 0$ contains three constants. But this number can be reduced to two independent constants by dividing all the terms by any one of the constants, provided it is not zero. Thus the equation can be written as

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0, \text{ if } C \neq 0 \text{ or, say, } lx + my + 1 = 0.$$

Thus any equation of a line is expected to contain only two independent constants. It geometrically means that a straight line can be completely determined by two independent conditions.

(6) To find (1) the intercepts on the co-ordinate axes, of the line $Ax + By + C = 0$, (2) the slope of the line and (3) the length and inclination of the perpendicular on it from the origin.

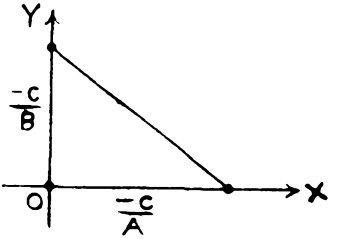


Fig. 5.14

(1) The intercepts can be obtained by considering the intersections of the line with the co-ordinate axes. Where the line cuts the x-axis, i. e. $y = 0$, we note that the ordinate of the point of intersection is zero and so the abscissa will be obtained by substituting the value of the ordinate y, viz. 0, in $Ax + By + C = 0$.

This gives $Ax + C = 0$, i. e. $x = -C/A$.

Similarly where the line cuts the y-axis, we have $x = 0$ and y is given by substituting this value of x viz. 0 in $Ax + By + C = 0$.

This gives $By + C = 0$, i. e. $y = -C/B$.

$\therefore$ the x-intercept $= -\frac{C}{A}$ and the y-intercept $= -\frac{C}{B}$.

The intercepts can also be obtained by writing the equation in the double-intercept form.

(2) The slope of the line $Ax + By + C = 0$ can be obtained by solving the equation for y.

Thus we have $y = -\frac{A}{B}x - \frac{C}{B}$.

Comparing it with the slope-intercept form, we see that

the slope of the line $= -\frac{A}{B} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

and that the intercept on the y axis, as we have already seen, is $-C/B$.

We may thus state the **Rule for writing the slope of any line :**

(i) Write the equation of the line so that all terms are on the Left Hand Side (R. H. S. being zero).

(ii) Slope of the line then $= -\frac{\text{coeff. of } x}{\text{coeff. of } y}$.

(3) The inclination and the length of the perpendicular from the origin to the line $Ax + By + C = 0$.

If $Ax + By + C = 0$ .. .. (i)

and $x \cos \alpha + y \sin \alpha = p$ .. .. (ii)

represent one and the same straight line, then the corresponding coefficients are proportional.

$$\therefore \quad \frac{\cos\alpha}{A} = \frac{\sin\alpha}{B} = \frac{-p}{C} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\pm\sqrt{A^2+B^2}}$$

$$= \frac{1}{\pm\sqrt{A^2+B^2}}.$$

$$\therefore \cos\alpha = \frac{A}{\pm\sqrt{A^2+B^2}}, \sin\alpha = \frac{B}{\pm\sqrt{A^2+B^2}}.$$

and $$p = \frac{-C}{\pm\sqrt{A^2+B^2}}.$$

Substituting these values in (ii) the normal form of line (i) is

$$\frac{A}{\pm\sqrt{A^2+B^2}}x + \frac{B}{\pm\sqrt{A^2+B^2}}y = \frac{-C}{\pm\sqrt{A^2+B^2}}. \quad \ldots \quad \ldots \quad (\text{iii})$$

Now sticking to our convention, introduced earlier for the sign of the perpendicular, we have to choose the proper sign for the radical in the denominator of the R. H. S. of (iii). Rewriting the equation (i), if required, so as to have a positive sign for the absolute term C and choosing the – ve sign for the radical, the equation (iii) can be written as

$$\frac{Ax}{-\sqrt{A^2+B^2}} + \frac{By}{-\sqrt{A^2+B^2}} = \frac{C}{\sqrt{A^2+B^2}}.$$

$\therefore$ by comparing it with the normal form

$$x\cos\alpha + \sin\alpha = p,$$

we deduce that the perpendicular distance, say, p when the term C has been so arranged as to have a positive sign, is given both in magnitude and sign by

$$p = \frac{C}{\sqrt{A^2+B^2}},$$

the inclination α of the perpendicular being determined by

$$\cos\alpha = \frac{A}{-\sqrt{A^2+B^2}} \quad \text{and} \quad \sin\alpha = \frac{B}{-\sqrt{A^2+B^2}}.$$

EXERCISE 5 (c)

(1) Find the intercepts on the axes made by the lines :
(i) $4x + 5y - 20 = 0$, (ii) $17x - 19y = 2$.

(2) Write the slopes of the following lines :
(i) $2x - 3y + 6 = 0$, (ii) $3x + 4y - 6 = 0$.

(3) Find the length and inclination of the perpendicular from the origin to the following lines :
(i) $3x + 4y - 12 = 0$, (ii) $5x - 12y + 26 = 0$, (iii) $40x - 9y = 41$.

(4) Interpret the equation $ax + by + c = 0$ in the following cases :
(i) $a = b = c = 0$, (ii) $a = b = 0$, $c \neq 0$, (iii) $a = c = 0$, $b \neq 0$,
(iv) $b = c = 0$, $a \neq 0$, (v) $a = 0$, $b \neq 0$, $c \neq 0$,
(vi) $b = 0$, $a \neq 0$, $c \neq 0$, (vii) $c = 0$, $a \neq 0$, $b \neq 0$.

(5) Taking the equation of a line, in the general form, $Ax + By + C = 0$, A, B, C not equal to zero, obtain the equation of a line (i) having intercepts a and b on the axes, (ii) at a distance p from the origin, the inclination of p being α, (iii) with a slope m and passing through the point (x_1, y_1), (iv) passing through the two points (x_1, y_1) and (x_2, y_2), (v) with a slope m and y-intercept $= c$.

[**Hint** : Eliminate A, B, C using the following hints :
(i) $a = -C/A$, $b = -C/B$, (ii) assuming $C > 0$, we have

$$A = \frac{-C\cos\alpha}{p}, \quad B = \frac{-C\sin\alpha}{p}, \quad \text{(iii)} \quad Ax_1 + By_1 + C = 0,$$

$m = -A/B$, (iv) $Ax_1 + By_1 + C = 0$, $Ax_2 + By_2 + C = 0$,
(v) $m = -A/B$, $c = -C/B$].

(6) Deduce the equation of a line in the normal form from the equation in the two intercepts form and conversely.
[**Hint** : $a = p \sec\alpha$, $b = p \operatorname{cosec}\alpha$, $\cos\alpha = p/a$, $\sin\alpha = p/b$.]

(7) A straight line intersects the x-axis in the point $A(a, 0)$ and the y-axis in $B(0, b)$. If $P(x, y)$ is any point on the line, obtain the equation of the line by using the fact that
area of $\triangle OAB$ = area of $\triangle OAP$ + area of $\triangle OPB$.

(8) Find the equations of the lines which have the slope $-3/4$ and which form with the co-ordinate axes a triangle of area 24 square units.

(9) Find the equation of the line passing through the point (1, 2) and having equal intercepts (both in magnitude and sign) on the co-ordinate axes.

(10) A straight line passes through the fixed point (2, 3). If it cuts the axes of co-ordinates in A and B, show that the locus of the mid-point of AB is $3x + 2y = 2xy$.

ANSWERS

(1) (i) 5, 4; (ii) 2/17, $-$ 2/19. (2) (i) 2/3, (ii) $-$ 3/4.
(3) (i) $p = 12/5$, $\alpha = \tan^{-1}(4/3)$, α acute.
(ii) $p = 2$, $\tan\alpha = -12/5$, α obtuse.
(iii) $p = 1$, $\tan\alpha = -9/40$, $3\pi/2 < \alpha < 2\pi$.

(4) (i) The equation does not exist, (ii) the intercepts $-c/a$, $-c/b$ are both infinite and hence there is no finite line; in fact there is no linear equation in x and y, (iii) the x-axis, (iv) the y-axis, (v) the line, $y = -c/b$, parallel to the x-axis, (vi) the line, $x = -c/a$, parallel to the y-axis, (vii) the line, $y = -ax/b$, passing through the origin.

(8) $3x + 4y + 24 = 0$, $3x + 4y - 24 = 0$. (9) $x + y - 3 = 0$.

(7) To find the ratio in which the line $ax + by + c = 0$ divides the segment joining the points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$.

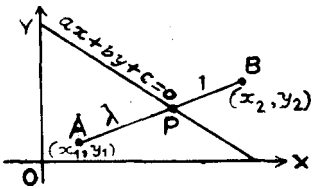


Fig. 5·15

Let $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ be any two points not lying on the line

$$ax + by + c = 0.$$

Let seg AB be divided by the line at a point P in the ratio $\lambda : 1$.

By the Section-Formula, we have

$$P \equiv \left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda} \right).$$

The co-ordinates of P, since it lies on the given line, satisfy its equation.

$$\therefore\ a\left(\frac{x_1 + \lambda x_2}{1 + \lambda} \right) + b\left(\frac{y_1 + \lambda y_2}{1 + \lambda} \right) + c = 0.$$

$$\therefore\ a(x_1 + \lambda x_2) + b(y_1 + \lambda y_2) + c(1 + \lambda) = 0.$$

$$\therefore\ \lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c) = 0.$$

$$\therefore\ \frac{\lambda}{1} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.$$

(8) The sign of the expression $ax + by + c$.

If $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign, then the ratio of division, in the last article is negative and hence P lies on seg AB produced and the points A, B lie on the same side of the line. However, if

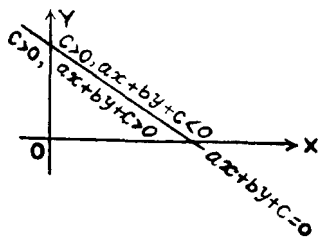


Fig. 5·16

$ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then the ratio of division, in the last article, is positive and hence P lies in between the points A and B and hence the points A and B lie on opposite sides of the line. Thus, the expression $ax_1 + by_1 + c$

and $ax_2 + by_2 + c$ are both + ve or both − ve when A, B are on the same side of the line $ax + by + c = 0$, but one + ve and the other − ve when the points A, B lie on opposite sides of the line.

We may, therefore, regard the line as dividing the plane into two half-planes; in one half-plane the expression $ax + by + c$ is + ve when for x, y we substitute the co-ordinates of any point in that half-plane; in the other half-plane, the expression is − ve. We may call these half-planes the + ve and − ve sides of the line. If the co-ordinates of the origin be substituted, the sign of the expression is the same as that of the constant term c in the equation. If the equation is, therefore, rewritten, if required, so as to have the constant term + ve, then the point (x_1, y_1) will lie on the origin side of the line if $ax_1 + by_1 + c$ is + ve and on the non-origin side of the line if $ax_1 + by_1 + c$ is − ve.

(9) Length of a perpendicular from a point to a line.

(i) The length of the perpendicular from the point $P(x_1, y_1)$ to the straight line $x \cos \alpha + y \sin \alpha = p$ is $\pm (p - x_1 \cos \alpha - y_1 \sin \alpha)$.

Let AB be the given line.

The length of the perpendicular from the origin on the line AB is p and α is the inclination of this perpendicular.

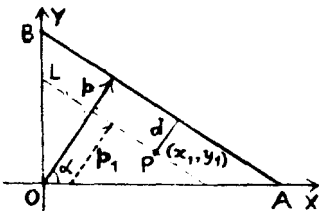


Fig. 5·17

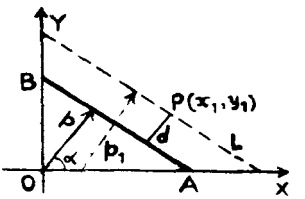


Fig. 5·18

Let p_1 be the length of the perpendicular from the origin on a line L drawn through the point $P(x_1, y_1)$ and parallel to AB.

The equation of the line L is $x \cos \alpha + y \sin \alpha = p_1$.

Since $P(x_1, y_1)$ lies on L, we have $x_1 \cos \alpha + y_1 \sin \alpha = p_1$.

From the Figure, it can be seen that the perpendicular distance 'd' from the point $P(x_1, y_1)$ to the line AB, is $p - p_1$ or $p_1 - p$ according as the point P and the origin lie on the same side or on opposite sides of the line AB.

$\therefore\ d = \pm (p - p_1) = \pm (p - x_1 \cos \alpha - y_1 \sin \alpha)$.

(ii) The length of the perpendicular from $P\ (x_1, y_1)$ to the line $ax + by + c = 0$ is $\pm \dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$.

Let $ax + by + c = 0$ and $x \cos \alpha + y \sin \alpha = p$ represent the same line.

We, then, have

$$\cos \alpha = \frac{a}{\pm \sqrt{a^2+b^2}}, \sin \alpha = \frac{b}{\pm \sqrt{a^2+b^2}}, \quad p = \frac{-c}{\pm \sqrt{a^2+b^2}}.$$

Now we know that the length of the perpendicular d from $P\ (x_1, y_1)$ to the line $x \cos \alpha + y \sin \alpha = p$ is

$$\pm (p - x_1 \cos \alpha - y_1 \sin \alpha).$$

Substituting the above values, we get

$$d = \pm \left[\frac{-c}{\pm \sqrt{a^2 + b^2}} - \frac{ax_1}{\pm \sqrt{a^2+b^2}} - \frac{by_1}{\pm \sqrt{a^2+b^2}} \right].$$

$$\therefore\ d = \pm \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

We get the ambiguous sign for d, the length of the perpendicular. The expression $ax_1 + by_1 + c$ also changes its sign as the point (x_1, y_1) moves from one side of the line to the other. It is, therefore, necessary to fix a convention for the sign of the perpendicular. We follow the convention that the perpendicular from the origin $(0, 0)$ on any line should be positive. The origin lies on the positive side of the line if we rewrite the equation of the line, if required, with its constant term positive. Thus, the perpendicular distance 'd' from the point (x_1, y_1) on the line $ax + by + c = 0$, where c is positive, is given, both in magnitude and sign, by

$$d = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

If the length of the perpendicular, thus obtained, has a positive sign, then the point (x_1, y_1) lies on the origin-side of the line: otherwise, it lies on the non-origin side of the line. If the length of the perpendicular is zero, the point evidently lies on the line.

Rule for obtaining the perpendicular distance from a given point to a given line.

(i) Write all the terms of the equation on the L. H. S. (R. H. S. being zero).

(ii) Rewrite the L. H. S., if required, so as to have a positive sign for the constant term.

(iii) Substitute for x, y the co-ordinates of the given point in the L. H. S. expression.

(iv) Divide the above expression by the positive square root of the sum of squares of the co-efficients of x and y.

Illustrative Examples.

(i) Find the perpendicular distance of the point $(-2, 3)$ to the line $x - y = 5$.

The equation of the line is $-x + y + 5 = 0$.

$$\therefore \text{ perpendicular distance } d = \frac{-(-2)+(3)+5}{\sqrt{(-1)^2+(1)^2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

(ii) Show that the points $(2, -1)$ and $(1, 1)$ are on opposite sides of the line $3x + 4y = 6$. Find which of the two points lies on the origin side of the line.

The equation of the line is $-3x - 4y + 6 = 0$.

Perpendicular from $(2, -1)$ on the line

$$= \frac{-3(2)-4(-1)+6}{\sqrt{(-3)^2+(-4)^2}} = \frac{4}{5}.$$

Perpendicular from $(1, 1)$ on the line

$$= \frac{-3(1)-4(1)+6}{\sqrt{(-3)^2+(-4)^2}}, \quad \text{i. e. } \frac{-1}{5}.$$

The signs of the perpendiculars are not the same.

Hence the points lie on opposite sides of the line.

The point $(2, -1)$ lies on the origin-side of the line.

(10) Perpendicular distance between two parallel lines.

The distance between two parallel lines is defined as the shortest (perpendicular) distance from any point on one of the lines to the other. In order to obtain the distance between two parallel lines, we take any convenient point on one of the lines and find the perpendicular distance from this point to the other line.

To show that the perpendicular distance between two parallel lines $L_1 : ax + by + c = 0$ and $L_2 : ax + by + c' = 0$ is

$$\frac{|c - c'|}{\sqrt{a^2 + b^2}}.$$

Let A be a point on L_1. Let the abscissa of the point A be zero. The ordinate of the point A can be obtained by putting $x = 0$ in the equation of the line L_1 and solving it for y.

Thus, when $x = 0$, we have $by + c = 0$, $\therefore y = -c/b$.

$\therefore$ the co-ordinates of the point A are $(0, -c/b)$.

Now the perpendicular distance from the point A to the line

$$L_2 \text{ is } = \frac{a(0) + b(-c/b) + c'}{\sqrt{a^2 + b^2}} = \frac{-c + c'}{\sqrt{a^2 + b^2}}.$$

Similarly, the perpendicular distance from a point, say, $(0, -c'/b)$ on L_2 to L_1 is

$$= \frac{a(0) + b(-c'/b) + c}{\sqrt{a^2 + b^2}} = \frac{c - c'}{\sqrt{a^2 + b^2}}.$$

In any case, the numerical magnitude of the perpendicular

$$\text{distance} = \frac{|c - c'|}{\sqrt{a^2 + b^2}}.$$

Before using this formula it is necessary to see that the coefficients of x and y in the equations of both the lines are equal both in magnitude and sign.

Example : Find the perpendicular distance between the parallel lines

(i) $3x + 2y - 6 = 0$, $3x + 2y - 12 = 0$;

(ii) $3x + 2y - 6 = 0$, $3x + 2y + 12 = 0$;

(iii) $y = mx + a$, $y = mx + b$.

Ans : (i) $6/\sqrt{13}$, (ii) $18/\sqrt{13}$, (iii) $\frac{|a - b|}{\sqrt{1 + m^2}}$.

(11) To find the measure of an angle between the two lines.

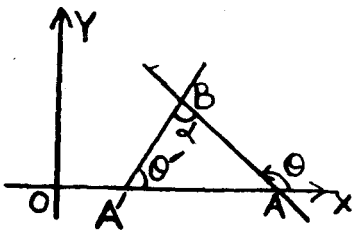


Fig. 5·19

$$y = mx + c \quad .. \quad .. \quad \text{(i)}$$

and $y = m'x + c'$. $\quad .. \quad .. \quad \text{(ii)}$

Let AB and $A'B$ be the two lines (i) and (ii) inclined to OX at angles θ and θ' so that $\tan\theta = m$ and $\tan\theta' = m'$.

If α is the angle between the two lines, then $\alpha = \theta - \theta'$.

$$\therefore \tan\alpha = \tan(\theta - \theta')$$

$$= \frac{\tan\theta - \tan\theta'}{1 + \tan\theta \tan\theta'}.$$

$$\therefore \tan\alpha = \frac{m - m'}{1 + mm'}. \qquad \text{(iii)}$$

When two lines intersect, they form two distinct angles which are supplementary. If the values of m and m' are such that the value of $\frac{m - m'}{1 + mm'}$ happens to be positive, then formula (iii), since the tangent ratio of an acute angle is positive, gives the acute angle between (i) and (ii) . However, if the values of m and m' are such that the value of $\frac{m - m'}{1 + mm'}$ happens to be negative, then formula (iii), since the tangent ratio of an obtuse angle is – ve, gives the obtuse angle between lines (i) and (ii). In any case, the acute angle between the two lines will be given by

$$\tan \alpha = \left| \frac{m - m'}{1 + mm'} \right| \quad \text{or} \quad \alpha = \tan^{-1} \left| \frac{m - m'}{1 + mm'} \right| . \quad .. \quad .. \quad (\text{iv})$$

It immediately follows from (iii) that the lines (i) and (ii) are parallel if $m = m'$ and are mutually perpendicular if $mm' = -1$.

Remarks. If the lines are given in the form

$$ax + by + c = 0 .. \quad (\text{v}), \qquad a'x + b'y + c' = 0 \quad .. \quad .. \quad (\text{vi}),$$

then their slopes are given by $\frac{-a}{b}$ and $\frac{-a'}{b'}$ respectively.

If α is the measure of an angle between these lines then from (iii)

$$\tan \alpha = \frac{\left(\frac{-a}{b}\right) - \left(\frac{-a'}{b'}\right)}{1 + \left(\frac{-a}{b}\right)\left(\frac{-a'}{b'}\right)} = \frac{a'b - ab'}{aa' + bb'} . \quad .. \quad .. \quad (\text{vii})$$

It immediately follows from (vii) that

(1) the acute angle α between (v) and (vi) is

$$\alpha = \tan^{-1} \left| \frac{a'b - ab'}{aa' + bb'} \right| ,$$

(2) the lines (v) and (vi) are parallel

if $a'b - ab' = 0$. i. e. if $\frac{a'}{a} = \frac{b'}{b} \neq \frac{c'}{c}$.

It will be seen that these lines are identical

if $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$.

(3) The lines (v) and (vi) are mutually $\perp$ if $aa' + bb' = 0$.

Note : When any one of the lines has an infinite slope, i. e. when the line coincides with the y-axis or is parallel to the y-axis then formula (iii) fails to give the result and we have to use formula (vii).

Example. Find the measure of the acute angle between the lines formed by

$$x = 2 \ldots\ldots (i), \quad x - \sqrt{3}y + 3\sqrt{3} = 0. \qquad \ldots (ii)$$

We use the formula (vii). Here we have $a = 1$, $b = 0$, $a' = 1$, $b' = -\sqrt{3}$.

$\therefore$ if α is the acute angle between the lines, then

$$\tan \alpha = \left| \frac{0 + \sqrt{3}}{1 + 0} \right| = \sqrt{3}.$$

$\therefore$ the measure of the acute angle between lines (i) and (ii) is $60°$.

(12) To find the point of intersection of two lines.

Let the lines be given by $ax + by + c = 0$, ... (i)

and $a'x + b'y + c' = 0$. .. .. (ii)

$\because$ the point of intersection lies on both the lines,

$\therefore$ the co-ordinates of the point of intersection of the two lines satisfy the equations of both the lines.

$\therefore$ The values of x, y obtained by solving equations (i) and (ii) simultaneously, give the required co-ordinates of the point of intersection.

Eliminating y, we get $x = \dfrac{bc' - b'c}{ab' - a'b}$.

Eliminating x, we get $y = \dfrac{ca' - c'a}{ab' - a'b}$.

$\therefore$ the point of intersection is $\left(\dfrac{bc' - b'c}{ab' - a'b}, \quad \dfrac{ca' - c'a}{ab' - a'b} \right)$.

For example, the point of intersection of the lines

$x - y - 1 = 0$ and $2x - 3y + 1 = 0$ is $(4, 3)$.

It may be verified that the co-ordinates $(4, 3)$ satisfy both the equations.

(13) To find the condition that three straight lines may be concurrent.

Let the three lines be

$$a_1 x + b_1 y + c_1 = 0, \quad .. \quad .. \quad (i)$$

$$a_2 x + b_2 y + c_2 = 0, \quad .. \quad .. \quad (ii)$$

$$a_3 x + b_3 y + c_3 = 0. \quad .. \quad .. \quad (iii)$$

The three straight lines will be concurrent if the point of intersection of any two of the lines lies on the third.

The point of intersection of (ii) and (iii) is

$$\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2} \quad , \quad \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2} \right), \quad \text{if } a_2 b_3 - a_3 b_2 \neq 0.$$

It lies on (i),

$$\text{if } a_1 \left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2} \right) + b_1 \left(\frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2} \right) + c_1 = 0.$$

$\therefore$ the given lines are concurrent if

$$a_1 (b_2c_3 - b_3c_2) + b_1 (c_2a_3 - c_3a_2) + c_1 (a_2b_3 - a_3b_2) = 0.$$

Note (1) : The equation of any line parallel to the line $ax + by + c = 0$ and passing through the point (x_1, y_1) is

$$a (x - x_1) + b (y - y_1) = 0.$$

Rule to write down the equation of a line parallel to a given line and passing through the given point (x_1, y_1).

(i) In the equation of the line write $(x - x_1)$ and $(y - y_1)$ for x and y respectively without altering their coefficients.

(ii) Drop the constant term.

Thus, the equation of a line passing through the point (2, 3) and parallel to the line $3x - 4y + 5 = 0$ is

$$3 (x - 2) - 4 (y - 3) = 0, \text{ i. e. } 3x - 4y + 6 = 0.$$

Note (2) : The equation of the line passing through the point (x_1, y_1) and perpendicular to the line $ax + by + c = 0$ is

$$b (x - x_1) - a (y - y_1) = 0.$$

Rule to write down the equation of the line perpendicular to a given line and passing through a given point (x_1, y_1).

(i) In the equation of the line, interchange the co-efficients of x and y and change the sign of any one of them.

(ii) Write $(x - x_1)$ and $(y - y_1)$ for x and y respectively.

(iii) Drop the constant term.

Thus, the equation of the line through the point (4, 5) and perpendicular to the line $3x - 2y + 5 = 0$ is

$$2 (x - 4) + 3 (y - 5) = 0, \text{ i. e. } 2x + 3y - 23 = 0.$$

Illustrative Examples.

Ex. 1. A straight line through the point $A (a, b)$ meets a line perpendicular to it and drawn through the point $B (c, d)$ in the point P. Find the equation of the locus of P and interepret it geometrically.

Let $P(x, y)$ be any point on the locus. By the condition of the problem

$$m(\angle APB) = 90°.$$

Hence, we have : slope of $AP \times$ slope of $BP = -1$.

$$\therefore \quad \frac{(y-b)}{(x-a)} \cdot \frac{(y-d)}{(x-c)} = -1.$$

$\therefore$ the equation to the locus is $(x-a)(x-c) + (y-b)(y-d) = 0$.

This locus can be seen to be a circle having seg AB as a diameter.

Ex. 2. Find the equations of the two lines passing through the point $(2, 3)$ and equally inclined to the co-ordinate axes.

The measures of inclinations of the lines being 45° and 135°, their equations are

$$\frac{y-3}{x-2} = \tan 45° \quad \text{and} \quad \frac{y-3}{x-2} = \tan 135°,$$

$$\text{i. e.,} \quad \frac{y-3}{x-2} = \pm 1, \quad \text{i. e.,} \quad y-3 \mp (x-2) = 0.$$

$\therefore$ The equations of the required lines are

$$x - y + 1 = 0 \quad \text{and} \quad x + y - 5 = 0.$$

Ex. 3. Find the equation of the line passing through the point $(-2, 3)$ and having intercepts equal in magnitude but opposite in sign on the co-ordinate axes.

The given point lies in the second quadrant. Let the possible line through the given point intersect the x-axis in the point $(-a, 0)$ and the y-axis in the point $(0, a)$.

$\therefore$ by the Two-Intercepts form, the equation of the line is

$$\frac{x}{-a} + \frac{y}{a} = 1, \quad \text{i. e.,} \quad -x + y = a.$$

The line is to pass through the point $(-2, 3)$.

$\therefore \quad -(-2) + 3 = a, \quad \therefore a = 5.$

$\therefore$ the equation of the required line is

$$-x + y = 5, \quad \text{i. e.,} \quad x - y + 5 = 0.$$

Ex. 4. If the line passing through the point $A(2, 3)$ and inclined at 60° to the x-axis meets the line $7x + 8y - 56 = 0$ in the point P, find the distance AP.

We use the parametric form of the equation of a line. The co-ordinates of a point lying on the given line and at a distance, say, $|r|$ from the given point A are

$$x = x_1 + r\cos\alpha, \quad y = y_1 + r\sin\alpha$$

where $(x_1, y_1) = (2, 3)$ and $\alpha = 60°$.

$$\therefore \quad x = 2 + r/2, \quad y = 3 + r\sqrt{3}/2.$$

Now this point is to lie on the line $7x + 8y - 56 = 0$.

$$\therefore \quad 7\left(2 + \frac{r}{2}\right) + 8\left(3 + \frac{r\sqrt{3}}{2}\right) - 56 = 0.$$

$$\therefore \quad r\left(\frac{7}{2} + 4\sqrt{3}\right) = 18.$$

$$\therefore \quad r = \frac{36}{7 + 8\sqrt{3}}, \quad \therefore AP = \frac{36}{7 + 8\sqrt{3}}.$$

Ex. 5. Find the locus of the mid-point of the portion of the variable line $x \cos \alpha + y \sin \alpha = p$, intercepted by the axes of co-ordinates, given that p remains constant.

In the equation of the line $x \cos \alpha + y \sin \alpha = p$, α is a variable parameter and as α varies, we get different straight lines. Let one such straight line intersect the x and y axes in A and B so that

$$A \equiv \left[\frac{p}{\cos \alpha}, 0\right] \text{ and } B \equiv \left[0, \frac{p}{\sin \alpha}\right].$$

[It is convenient here to take the co-ordinates of the point belonging to the locus as (x', y') to avoid confusion with the current co-ordinates (x, y) which occur in the equation of the given data. Ultimately the equation of the locus can be written by changing (x', y') to (x, y)].

Let $P(x', y')$ be the mid-point of seg AB.

$$\therefore \quad x' = \frac{p}{2 \cos \alpha}, \quad y' = \frac{p}{2 \sin \alpha}.$$

$$\therefore \quad \cos \alpha = \frac{p}{2x'}, \quad \sin \alpha = \frac{p}{2y'}.$$

The parameter α can be eliminated by using the trigonometrical identity $\cos^2 \alpha + \sin^2 \alpha = 1$.

$$\therefore \quad \frac{p^2}{4x'^2} + \frac{p^2}{4y'^2} = 1.$$

$\therefore$ (x', y') always satisfies the equation $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$.

$\therefore$ the locus of the mid-point is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$.

* **Ex. 6.** Find the co-ordinates of the orthocentre of the triangle formed by the lines

$$x - y - 5 = 0,\ 2x - y - 8 = 0 \text{ and } 3x - y - 9 = 0.$$

Let ABC be the triangle formed by the given lines.

Let $AB \equiv x - y - 5 = 0$... (i), $BC \equiv 2x - y - 8 = 0$... (ii) and $CA \equiv 3x - y - 9 = 0$... (iii). Solving equations (i) and (ii), (ii) and (iii), (iii) and (i) we get the co-ordinates of the vertices of the $\triangle ABC$. We have $B = (3, -2)$, $C = (1, -6)$ and $A = (2, -3)$.

The orthocentre is the point of concurrence of the three altitudes of the triangle. It is sufficient to find the equations of two of the altitudes to determine the orthocentre. Let AD and BE be the two altitudes.

Now AD is a straight line drawn through the point $A(2, -3)$ and perpendicular to $BC \equiv 2x - y - 8 = 0$.

$\therefore$ equation of AD is

$$(x - 2) + 2(y + 3) = 0, \text{ i.e., } x + 2y + 4 = 0. \quad \ldots \text{ (iv)}$$

Similarly BE is a straight line drawn through the point $B\,(3, -2)$ and perpendicular to $CA \equiv 3x - y - 9 = 0$.

$\therefore$ equation of BE is

$$(x - 3) + 3(y + 2) = 0, \quad \text{i. e.,} \quad x + 3y + 3 = 0. \qquad \ldots\ (v)$$

Solving equations (iv) and (v), we get the co-ordinates of the orthocentre as $(-6, 1)$.

Ex. 7. Find the equations of the two straight lines through the point $(2, -1)$ making an angle of $45°$ with the line $6x + 5y + 1 = 0$.

Let θ be the measure of inclination of the given line so that $\tan\theta = -6/5$.

The measures of inclinations of the two required lines are

$$\theta + \frac{\pi}{4} \quad \text{and} \quad \pi - \frac{\pi}{4} + \theta.$$

$\therefore$ The slopes of the two required lines are

$$\tan\left(\theta + \frac{\pi}{4}\right) \text{ and } \tan\left(\pi - \frac{\pi}{4} + \theta\right) = \tan\left(\theta - \frac{\pi}{4}\right),$$

$$\text{i. e.} \quad \frac{\tan\theta \pm \tan\pi/4}{1 \mp \tan\theta \cdot \tan\pi/4}, \quad \text{i. e.} \quad \frac{-\frac{6}{5} \pm 1}{1 \mp \left(-\frac{6}{5}\right)\cdot 1} = \frac{-6 \pm 5}{5 \mp (-6)}.$$

Taking the upper signs, the slope of one line $= -1/11$. Taking the lower signs, the slope of the second line $= 11$.

$\therefore$ the required lines are

$$y + 1 = -\frac{1}{11}(x - 2) \quad \text{and} \quad y + 1 = 11(x - 2),$$

$$\text{i. e.,} \quad x + 11y + 9 = 0 \quad \text{and} \quad 11x - y - 23 = 0.$$

***Ex. 8.** Find the equations of the bisectors of the angles between the lines $12x - 5y + 7 = 0$ and $3x - 4y - 12 = 0$. State which bisector bisects the angle containing the origin and which bisector bisects the acute angle between the lines.

If (x, y) is any point on either bisector, then it is equidistant from the two given lines. Hence the equations of the bisectors are

$$\frac{12x - 5y + 7}{\sqrt{144 + 25}} = \pm\frac{3x - 4y - 12}{\sqrt{9 + 16}}.$$

These equations, after simplification, become

$$9x - 7y - 11 = 0 \qquad \ldots \qquad \ldots\ (i)$$

and

$$21x + 27y + 191 = 0. \qquad \ldots \qquad \ldots\ (ii)$$

The bisector of the angle containing the origin is

$$\frac{12x - 5y + 7}{13} = \frac{-3x + 4y + 12}{5}.$$

[since every point on this bisector lies either on (i) the origin side or (ii) the non-origin side of both the lines]

i. e. $\quad 5(12x - 5y + 7) - 13(-3x + 4y + 12) = 0,$

i. e. $\quad (60 + 39)x - (25 + 52)y + 35 - 156 = 0,$

i. e. $\quad 99x - 77y - 121 = 0,$

i. e. $\quad 9x - 7y - 11 = 0.$

Let ϕ be the angle between the line $L_1 : 12x-5y+7=0$ and the bisector, say $B_1 : 9x-7y-11=0$.

$$\therefore \text{ we have } \tan\phi = \frac{\frac{12}{5}-\frac{9}{7}}{1+\frac{12}{5}\cdot\frac{9}{7}} = \frac{84-45}{35+108} = \frac{39}{143}.$$

We observe that $|\tan\phi| < 1$.

$\therefore$ The bisector $B_1 : 9x-7y-11=0$ must be the acute angle bisector.

Ex. 9. Find the incentre of the triangle formed by the lines $x+1=0$, $3x-4y=5$ and $5x+12y=27$.

Draw a rough sketch of the triangle as shown in the figure. We observe that the origin lies inside the triangle ABC. Let I be the incentre of the triangle. The incentre is the point of concurrence of the three internal bisectors of the angles of a triangle. Let us write the equations of the sides having a positive sign for the constant terms.

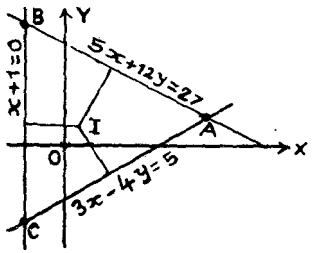


Fig. 5·20

We have the line $AB \equiv -5x-12y+27=0$,

the line $BC \equiv x+1=0$,

the line $CA \equiv -3x+4y+5=0$.

The internal bisector of $\angle B$ is

$$\frac{-5x-12y+27}{13} = +(x+1), \text{ i.e., } 18x+12y-14=0. \quad \ldots\ \ldots\ (i)$$

The internal bisector of $\angle C$ is

$$\frac{-3x+4y+5}{5} = +(x+1), \text{ i.e., } 8x-4y=0. \quad \ldots\ \ldots\ (ii)$$

Solving equations (i) and (ii), we get the co-ordinates of the incentre as $I = (1/3, 2/3)$.

EXERCISE 5 (d)

(1) The segment joining the points $P(a, b)$ and $Q(c, d)$ is divided into n equal parts. Show that the co-ordinates of the rth point of subdivision are

$$\left[a+\frac{r(c-a)}{n},\ b+\frac{r(d-b)}{n}\right].$$

[**Hint** : The rth point of subdivision divides seg PQ in the ratio $r : (n-r)$.]

(2) Find the co-ordinates of the circumcentre O, the centroid G and the orthocentre P of the triangle whose vertices are $(-2, 3)$, $(2, -1)$ and $(4, 0)$.

[**Hint** : G divides OP internally in the ratio $1 : 2$].

(3) Find the equation of a straight line passing through the point $(2, 3)$ if its intercept on the x-axis is twice its intercept on the y-axis.

(4) A straight line cuts off on the axes of co-ordinates, positive intercepts whose sum is 7. If the line passes through the point $(-3, 8)$, find its equation.

(5) Find the equation of the line joining the points $(7, -5)$ and $(-4, 3)$. Show that the point $(-15, 11)$ lies on this line.

(6) Show that the medians of the triangle whose vertices are $(1, 0)$, $(0, 1)$ and $(2, 3)$ are given by

$$x - 1 = 0, \quad 3y - x = 3 \quad \text{and} \quad 3y - 5x + 1 = 0.$$

(7) If the straight line through the point $P(x_1, y_1)$ making an angle α with the axis of x meets the line $Ax + By + C = 0$ at Q, prove that

$$PQ = \left| \frac{Ax_1 + By_1 + C}{A \cos \alpha + B \sin \alpha} \right|.$$

(8) Find the ratio in which the line segment from the point $(2, 1)$ to $(5, 2)$ is divided by the line passing through the points $(3, 2)$ and $(5, 0)$.

(9) A straight line through a given point (h, k) cuts the co-ordinate axes in A and B. Show that the locus of the mid-point of seg AB is

$$kx + hy - 2xy = 0.$$

[**Hint** : Let $P(x_1, y_1)$ be the mid-point of seg AB. We have x-intercept $= 2x_1$ and y-intercept $= 2y_1$. Write the equation of the line in the intercepts form and use the fact that (h, k) lies on it.]

(10) Find the equation of the line which bisects the join of $(9, 2)$ and $(4, -7)$ and is perpendicular to the join of $(3, 8)$ and $(5, 4)$.

(11) Find the co-ordinates of the foot of the perpendicular drawn from the point $(1, 2)$ to the straight line $x - 3y + 7 = 0$.

(12) A triangle is formed by the three lines $x + y = 1$, $3x - y = 7$ and $3y = x + 3$. Show that the co-ordinates of the circumcentre of the triangle are $(7/4, 3/4)$.

(13) Find the co-ordinates of the orthocentre of the triangle whose angular points are at $(1, 0)$, $(2, -4)$ and $(-5, -2)$.

(14) The equations of two lines are $2x + 3y + 5 = 0$ and $4x - 5y - 23 = 0$. Find the co-ordinates of the point of intersection of these lines. Show that these two lines and the line $2x - y - 7 = 0$ are concurrent.

(15) The co-ordinates of the vertex A of a square $ABCD$ are $(1, 2)$ and the equation of the diagonal BD is $x + 2y = 10$. Find the equation of the other diagonal and the co-ordinates of the centre of the square.

(16) $(-2, 5)$ and $(6, 7)$ are the extremities of a diagonal of a rhombus and $x = 2y - 12$ is the equation of a side. Determine the equations of the other sides and the other diagonal of the rhombus.

(17) Find the equations of the lines making an angle of $45°$ with
(i) the line $2x - y + 7 = 0$ and passing through the point $(4, 5)$,
(ii) the line $3x - y + 5 = 0$ and passing through the point $(2, 3)$.

(18) Find the two points on the
(i) x-axis, which are at a distance 2 from $3x - 4y - 6 = 0$;
(ii) line $2x + 3y + 4 = 0$, which are at a distance 2 from the line $3x + 4y - 6 = 0$.

(19) **Find the equations of the lines parallel to the line $12x - 5y - 15 = 0$ and at a perpendicular distance from it numerically equal to 4 units.**

[**Hint** : Let $P(x_1, y_1)$ be any point on the required line.

We have $\pm \dfrac{12x_1 - 5y_1 - 15}{13} = 4.$]

(20) Find the equations of the lines through the point $(4, -2)$ and at a perpendicular distance of 2 units from the origin.

[**Hint** : The required line has the equation

$y + 2 = m(x - 4)$, i. e., $mx - y - (4m + 2) = 0$, say.

We have $\pm \dfrac{4m + 2}{\sqrt{m^2 + 1}} = 2$. This gives $m = 0$ or $-4/3$.]

(21) Show that the lines $x = 5$, $y = 5$, $5x + 12y = 65$ all touch a circle whose centre is at the origin.

(22) Two vertices of a triangle are at the points $A(3, -1)$, $B(-2, 3)$ and the orthocentre is at the origin. Find the co-ordinates of the remaining vertex.

[**Hint** : The third vertex C will lie on the line, through the origin O, perpendicular to AB and also on the line through B, perpendicular to AO, since the orthocentre is the point of concurrence of the three altitudes of a triangle.]

(23) Find the in-centre of the triangle formed by the lines

(i) $x - y + 2 = 0$, $y = 7x - 10$, $41x - y + 58 = 0$;

(ii) $x = 0$, $y = 0$, $4x - 3y = 12$.

(24) Find the area of the parallelogram formed by the lines

(i) $4y - 3x - a = 0$, $3y - 4x + a = 0$, $4y - 3x - 3a = 0$, $3y - 4x + 2a = 0$.

(ii) $3y - 2x - a = 0$, $2y - 3x + a = 0$, $2x - 3y + 3a = 0$, $3x - 2y - 2a = 0$.

[**Hint** : Area of a parallelogram is pp' cosec α where p and p' are perpendicular distances between the pairs of parallel sides and α is the angle between a pair of its adjacent sides.]

(25) Find the equations of the bisectors of the angles between the lines

(i) $2x - 3y + 7 = 0$, $3x = 2y - 5$;

(ii) $3x + 4y - 7 = 0$, $5x - 12y + 5 = 0$.

In each case state which bisector bisects the angle containing the origin and which bisector bisects the acute angle between the lines.

(26) Fill in the blanks in the following :

(i) The equations of the lines throught the point $(3, -2)$ and (a) parallel to the x-axis, (b) parallel to the y-axis are respectively ..... and .....

(ii) The slopes of the lines $3x + 7y = 10$ and $2x - 5y = 6$ are respectively .....and.....

(iii) The lines $2x - 3y = 7$ and $x + cy = 10$ are (a) parallel to each other if, c = ... (b) mutually perpendicular if, c = ...

(iv) The equation of the line making intercepts 7 and -5 on the x and y axes respectively is.....

(v) The points $(3, 2)$ and $(-4, 3)$ lie on .....sides of the line $2x + 5y - 8 = 0$

(vi) The points $(2, 0)$ and $(0, 7)$ lie on .....side of the line $3x + y - 9 = 0$

(vii) The perpendicular distance between the line $2x + 3y - 7 = 0$ and $4x + 6y + 9 = 0$ is.....

(viii) The lines $x - y - 7 = 0$, $2x + 3y - 4 = 0$ and $x + ky = 9$ are concurrent if $k =$ .....

(ix) The acute angle between the lines $2x + 3y = 10$ and $3x - 7y = 5$ is $\tan^{-1}$ (.....).

(x) The co-ordinates of the image of the origin in the line $x + y - 4 = 0$ are (....., ...).

(xi) The co-ordinates of the point in which the line $x + y - 5 = 0$ meets the locus $(x - 2)^2 + (y - 3)^2 = 8$ are (....., ...) and (.....).

(xii) The equations of the bisectors of the angles between the lines $x \cos\theta + y \sin\theta - 2 = 0$ and $x \sin\theta + y \cos\theta + 2 = 0$ are ....., ...

ANSWERS

(2) $O \equiv (3/2, 5/2)$, $G \equiv (4/3, 2/3)$, $P \equiv (1, -3)$. (3) $x + 2y - 8 = 0$.

(4) $4x + 3y - 12 = 0$. (5) $8x + 11y - 1 = 0$. (8) 1 : 1.

(10) $2x - 4y - 23 = 0$. (11) $(4/5, 13/5)$. (13) $(11/13, -7/13)$.

(14) $(2, -3)$. (15) $y = 2x$, $(2, 4)$.

(16) $x - 2y + 8 = 0$, $x - 38y + 260 = 0$, $x - 38y + 192 = 0$, $4x + y - 14 = 0$.

(17) (i) $3x + y - 17 = 0$, $x - 3y + 11 = 0$; (ii) $x - 2y + 4 = 0$; $2x + y - 7 = 0$.

(18) (i) $(16/3, 0)$, $(-4/3, 0)$; (ii) $(64, -44)$, $(4, -4)$.

(19) $12x - 5y - 67 = 0$, $12x - 5y + 37 = 0$.

(20) $y + 2 = 0$, $4x + 3y = 10$. (22) $(-36/7, -45/7)$.

(23) (i) $(0, 0)$, (ii) $(1, -1)$. (24) (i) $2a^2/7$, (ii) $2a^2/5$.

(25) (i) $x + y - 2 = 0$, $5x - 5y + 12 = 0$;

(ii) $32x - 4y - 33 = 0$, $7x + 56y - 58 = 0$.

In each answer the first bisector bisects the angle containing the origin and the second one bisects the acute angle between the lines.

(26) (i) (a) $y = -2$, (b) $x = 3$; (ii) $-3/7$, $2/5$; (iii) (a) $-3/2$, (b) $2/3$;

(iv) $5x - 7y = 35$; (v) opposite; (vi) same, (vii) $23/2\sqrt{13}$; (viii) -2;

(ix) $23/15$; (x) $(4, 4)$; (xi) $(0, 5)$, $(4, 1)$;

(xii) $x + y = 0$; $(x - y)(\cos\theta - \sin\theta) = 4$.

□ □ □

CHAPTER 6

Formulae for Compound Angles

(1) Trigonometrical Ratios of $-\theta$ in terms of those of θ.

To prove that for any angle θ,

$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta.$$

Let P and P' be two points on the unit circle such that $m(\angle XOP) = \theta$ and $m(\angle XOP') = -\theta$.

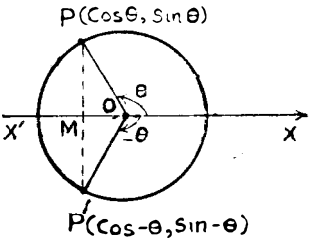


Fig. 6·1

$\therefore$ $P = (\cos\theta, \sin\theta)$

and $P' = [\cos(-\theta), \sin(-\theta)]$.

$\because$ P and P' are symmetrically placed with respect to the x-axis,

$\therefore$ the x-co-ordinates of P and P' are the same and their y-co-ordinates are equal in magnitude but opposite in sign.

$\therefore$ $\cos(-\theta) = \cos\theta$ .... (i), and $\sin(-\theta) = -\sin\theta$; ....(ii)

$$\therefore \quad \tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta;$$

$$\text{and} \quad \sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta;$$

similarly, $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$; and $\cot(-\theta) = -\cot\theta$.

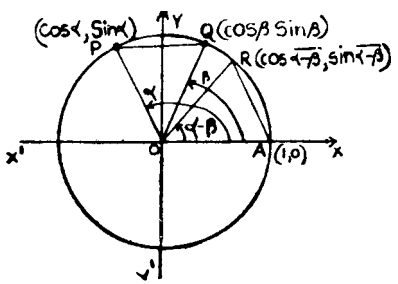


Fig. 6·2

(2) Addition and Subtraction Formulae for the Cosine :

To prove that for any angles α, β

$\cos(\alpha - \beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Because of periodicity we can assume that $\alpha, \beta \in [0, 2\pi)$. Further since

$\cos(\alpha - \beta) = \cos(\beta - \alpha)$,

[as $\cos(-\theta) = \cos\theta$]

we assume for the sake of convenience that $\alpha > \beta$. Thus, we take $0 \leqslant \beta < \alpha < 2\pi$.

Let the points P, Q on the unit circle, with O as centre, represent the numbers α, β respectively so that

$$P = (\cos\alpha, \sin\alpha), \quad Q = (\cos\beta, \sin\beta).$$

We have $l(\text{arc } QP) = l(\text{arc } AP) - l(\text{arc } AQ) = \alpha - \beta$.

Now let R be the point on the unit circle such that

$$l(\text{arc } AR) = \overline{\alpha - \beta} \text{ so that } R = (\cos\overline{\alpha - \beta}, \sin\overline{\alpha - \beta}).$$

Further as $l(\text{arc } AR) = l(\text{arc } QP)$,

$\therefore$ $l(\text{chord } AR) = l(\text{chord } QP)$.

$\therefore$ $[l(\text{chord } AR)]^2 = [l(\text{chord } QP)]^2$.

$\therefore$ using the distance formula we get

$$[\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta) - 0]^2$$
$$= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2.$$

$\therefore$ using the identity $\cos^2\theta + \sin^2\theta = 1$, we get

$$2 - 2\cos(\alpha-\beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

$\therefore$ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. ... ... (I)

Next replacing β by $-\beta$ we get

$$\cos(\alpha+\beta) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta),$$

i. e., $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. ... ... (II)

Note : Denoting $(\alpha-\beta)$ by θ, we have from the above figure

$$[l(\text{chord } AR)]^2 = (\cos\theta - 1)^2 + (\sin\theta - 0)^2$$
$$= 1 + \cos^2\theta - 2\cos\theta + \sin^2\theta;$$

$\therefore$ $[l(\text{chord } AR)]^2 = 2(1 - \cos\theta)$, where θ is the angle subtended by the chord of the circle at the centre of the unit circle. If the circle be of radius r we shall have

$$[l(\text{chord})]^2 = 2r^2(1 - \cos\theta).$$

(3) **Trigonometrical Ratios of $\left(\frac{\pi}{2} \pm \theta\right)$ in terms of those of θ.**

By using (I) of article 2, we have

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta;$$

but $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$.

$\therefore$ we have $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. ... (i)

Similarly from (I) of article 2, we have

$$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \cos\frac{\pi}{2}\cos\left(\frac{\pi}{2} - \theta\right) + \sin\frac{\pi}{2}\sin\left(\frac{\pi}{2} - \theta\right)$$

$\therefore \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$, i. e., $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ · ... (ii)

Similarly from (II) of article 2, we can get

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \cdot \qquad \ldots \text{ (iii)}$$

Similarly, we get $\cos\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + \theta\right)\right]$

$= \cos\frac{\pi}{2}\cos\left(\frac{\pi}{2} + \theta\right) - \sin\frac{\pi}{2}\sin\left(\frac{\pi}{2} + \theta\right)$

i. e., $\cos(\pi + \theta) = -\sin\left(\frac{\pi}{2} + \theta\right)$;

$\therefore \sin\left(\frac{\pi}{2} + \theta\right) = -[\cos\pi\cos\theta - \sin\pi\sin\theta]$;

i. e., $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta.$ ... (iv)

(4) **Addition and Subtraction Formulae for the Sine.**

In (II) of article 2, replace β by $\left(\frac{\pi}{2} - \beta\right)$; we then get

$$\cos\left(\alpha + \frac{\pi}{2} - \beta\right)$$

$$= \cos\alpha\cos\left(\frac{\pi}{2} - \beta\right) - \sin\alpha\sin\left(\frac{\pi}{2} - \beta\right);$$

$\therefore \cos\left[\frac{\pi}{2} - \left(\beta - \alpha\right)\right] = \cos\alpha\sin\beta - \sin\alpha\cos\beta.$

$\therefore \sin(\beta - \alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha.$

$\therefore \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$ ..... (III)

Similarly, replacing β by $\left(\frac{\pi}{2} - \beta\right)$ in (I) of article 2, we get

$$\cos\left[\alpha - \left(\frac{\pi}{2} - \beta\right)\right] = \cos\alpha\cos\left(\frac{\pi}{2} - \beta\right) + \sin\alpha\sin\left(\frac{\pi}{2} - \beta\right)$$

which on simplification gives

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \quad \ldots\ldots\ldots \quad (IV)$$

(I) and (III) are known as the **subtraction formulae** and (II) and (IV) as the **addition formulae** for the cosine and the sine respectively.

(5) **Relations between the trigonometrial ratios of $\pi \pm \theta$ in terms of those of θ.**

We have already obtained the ratios of $\frac{\pi}{2} \pm \theta$ in terms of those of θ. By making use of the addition and subtraction formulae for the sine and the cosine we can easily connect the trigonometric ratios of $\pi \pm \theta$ with those of θ.

Thus $\sin(\pi \pm \theta) = \sin\pi\cos\theta \pm \cos\pi\sin\theta$

$= \mp \sin\theta$, since $\sin\pi = 0$, $\cos\pi = -1$.

Hence $\sin(\pi + \theta) = -\sin\theta$, $\sin(\pi - \theta) = \sin\theta$.

Similarly we shall get $\cos(\pi + \theta) = -\cos\theta$, $\cos(\pi - \theta) = -\cos\theta$.

The other trigonometrical ratios can now be deduced in terms of those of θ.

Remarks. The results obtained in articles 1,3 and 5 can be summed up as follows.

(i) Naming sine, cosine; tangent, cotangent; and sec, cosec; as the co-ratios of each other, we have the result :

any ratio of an angle = co-ratio of its complement.

(ii) Naming θ and $-\theta$ as opposite angles we have the result :

The cosines of opposite angles are equal in magnitude and sign, while the sines of opposite angles are equal in magnitude but have opposite signs.

(iii) The sines of two supplementary angles are equal in magnitude and sign, while the cosines of two supplementary angles are equal in magnitude but have opposite signs.

(6) **Allied Angles.** The angles $-\theta$ and θ, $\frac{\pi}{2} \pm \theta$ and θ, and $\pi \pm \theta$ and θ are generally referred to as allied angles. Relations between the trigonometrical ratios of allied angles obtained in articles 1, 3 and 5 are given in the following table for immediate reference. Students are advised to remember these relations.

Angle → / Ratio ↓	$-\theta$	$\frac{\pi}{2} - \theta$	$\frac{\pi}{2} + \theta$	$\pi - \theta$	$\pi + \theta$
Cosine	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
sine	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

The ratios sec, cosec and cot have not been tabulated since their values can be written down as they are only the reciprocals of cosine, sine and tangent respectively.

(7) **Trigonometrical Ratios of any angle in terms of those of a positive acute angle.** This is possible by using the following results :

(i) The trigonometrical ratios of negative angles can be expressed in terms of those of positive angles. (§ 1)

(ii) Due to periodicity the trigonometrical ratios of angles $> 360°$ can be obtained in terms of those of angles lying between $0°$ and $360°$.

(iii) By using the rules for the ratios of supplementary and complementary angles the ratios of any angle between $0°$ and $360°$ can be expressed in terms of those of a positive acute angle.

Illustrative Examples.

Ex. 1. Find (i) $\sin(-1020°)$, (ii) $\cos(-17\pi/6)$, (iii) $\tan(14\pi/3)$ in terms of the trigonometrical ratios of a positive acute angle and hence find their values.

(i) $\sin(-1020°) = -\sin(1020°)$

$= -\sin(2 \times 360° + 300°)$

$= -\sin 300° = -\sin(360° - 60°)$

$= -\sin(-60°) = \sin 60° = \frac{\sqrt{3}}{2}$.

(ii) $\cos\left(-\frac{17\pi}{6}\right) = \cos\left(\frac{17\pi}{6}\right) = \cos\left(2\pi + \frac{5\pi}{6}\right)$

$= \cos\left(\frac{5\pi}{6}\right) = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$.

(iii) $\tan\left(\frac{14\pi}{3}\right) = \tan\left(4\pi + \frac{2\pi}{3}\right) = \tan\left(\frac{2\pi}{3}\right)$

$= \tan\left(\pi - \frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}.$

Ex. 2. Prove that

(i) $\tan^2(180^\circ + \theta) + \sin^2(180^\circ + \theta) + \cos^2(180^\circ + \theta) = \sec^2\theta.$

(ii) $\sin 10^\circ + \cot 35^\circ + \tan 55^\circ = 2\cot 35^\circ + \cos 80^\circ.$

(i) L. H. S. $= \tan^2\theta + (-\sin\theta)^2 + (-\cos\theta)^2$

$= \tan^2\theta + \sin^2\theta + \cos^2\theta$

$= \tan^2\theta + 1 = \sec^2\theta =$ R. H. S.

(ii) L. H. S. $= \sin 10^\circ + \cot 35^\circ + \tan(90^\circ - 35^\circ)$

$= \sin 10^\circ + \cot 35^\circ + \cot 35^\circ$

$= \cos 80^\circ + 2\cot 35^\circ =$ R. H. S.

EXERCISE 6 (a)

(1) Find the sines and cosines of the following angles.

(i) $5\pi/3$, (ii) $43\pi/6$, (iii) $-65\pi/6$.

(2) Express in terms of trigonometrical ratios of positive acute angles the following.

(i) $\cos 1419^\circ$, (ii) $\sec 1013^\circ$, (iii) $\cot(-2308^\circ)$.

(3) Prove that

$$\frac{\sin(90^\circ - \theta)}{\sin\theta} + \frac{\operatorname{cosec}(90^\circ - \theta)}{\operatorname{cosec}\theta} = \sec\theta\operatorname{cosec}\theta.$$

(4) Find the value of $\cos 135^\circ \sin 120^\circ - \cos 225^\circ \sin 300^\circ$.

(5) Show that

$$\sec(2\pi - \theta)\sin(\pi/2 + \theta) - \cos(3\pi/2 + \theta)\sin(\pi - \theta) = \cos^2\theta.$$

(6) Show that

$$\operatorname{cosec}(90^\circ - \theta)\operatorname{cosec}(90^\circ + \theta) + \sec(90^\circ - \theta)\sec(90^\circ + \theta) = \tan^2\theta - \cot^2\theta.$$

(7) Show that

$$\frac{\sin 20^\circ \cos^2 70^\circ + \cos 20^\circ \sin^2 70^\circ}{\sin 20^\circ + \cos 20^\circ} = 1 - \sin 70^\circ \cos 70^\circ.$$

(8) Prove that

$$[1 + 2\cos^2(180^\circ - \theta)] = \sin^2\theta\,[1 + 3\tan^2(90^\circ + \theta)].$$

(9) Show that

$$\cot^2(\pi + A) - \cos^2(\pi - A) = [\sin(\pi - A) + \operatorname{cosec}(2\pi - A)]^2.$$

ANSWERS

(1) (i) $-\sqrt{3}/2$, $1/2$, (ii) $-1/2$, $-\sqrt{3}/2$, (iii) $-1/2$, $-\sqrt{3}/2$.

(2) (i) $\cos 21^\circ$, (ii) $\operatorname{cosec} 23^\circ$, (iii) $\tan 58^\circ$. (4) $-\sqrt{6}/2$.

(8) Addition and Subtraction Formulae for the tangent.

To prove that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}, \quad \ldots (1)$$

and $$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}. \quad \ldots (2)$$

We have $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta},$$ on dividing

both the numerator and the denominator by $\cos\alpha\cos\beta$.

Similarly, $\tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

$$= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$ on dividing both the

numerator and the denominator by $\cos\alpha\cos\beta$.

Alternatively, we can obtain (2) from (1) by replacing β by $-\beta$.

(9) Formulae for the conversion of sums into products.

Replacing α and β by A and B respectively in articles 2 and 4 we get

$$\cos(A-B) = \cos A\cos B + \sin A\sin B. \quad \ldots (i)$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B. \quad \ldots (ii)$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B. \quad \ldots (iii)$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B. \quad \ldots (iv)$$

If we write $A+B=C$ and $A-B=D$

so that $$A = \frac{C+D}{2}, \quad B = \frac{C-D}{2},$$

we have by adding (ii) and (i)

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}, \quad \ldots (v)$$

and by subtracting (i) from (ii) we have

$$\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \quad \ldots \text{(vi)}$$

$$= 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}. \quad \ldots \text{(vii)}$$

Similarly, from (iii) and (iv) we get

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}, \quad \ldots \text{(viii)}$$

and $$\sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}. \quad \ldots \text{(ix)}$$

(10) **Formulae for conversion of products into sums.**

By addition and subtraction of (i) and (ii) in article 9, we get

$$2 \cos A \cos B = \cos (A-B) + \cos (A+B) \quad \ldots \text{(x)}$$

and $$2 \sin A \sin B = \cos (A-B) - \cos (A+B). \quad \ldots \text{(xi)}$$

Similarly by addition and subtraction of (iii) and (iv) in article 9 we have

$$2 \sin A \cos B = \sin (A+B) + \sin (A-B). \quad \ldots \text{(xii)}$$

and $$2 \cos A \sin B = \sin (A+B) - \sin (A-B). \quad \ldots \text{(xiii)}$$

(11) **Multiple angle Formulae.**

(a) Double Angle Formulae.

(1) We have $\sin 2A = \sin (A+A)$

$$= \sin A \cos A + \cos A \sin A = 2 \sin A \cos A \quad \ldots \text{(i)}$$

$$= \frac{2 \sin A \cos A}{\sin^2 A + \cos^2 A} = \frac{2 \tan A}{1 + \tan^2 A}, \quad [\text{ on dividing each term by } \cos^2 A]$$

$$= \frac{2t}{1+t^2}, \text{ where } t = \tan A. \quad \ldots \text{(ii)}$$

(2) We have $\cos 2A = \cos (A+A)$

$$= \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A \quad \ldots \text{(i)}$$

$$= 2 \cos^2 A - 1 \quad \ldots \text{(ii)}$$

$$= 1 - 2 \sin^2 A. \quad \ldots \text{(iii)}$$

Again $\cos 2A = \dfrac{\cos^2 A - \sin^2 A}{\sin^2 A + \cos^2 A}$

$= \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$ [on dividing each term by $\cos^2 A$]

$= \dfrac{1 - t^2}{1 + t^2}$, where $t = \tan A$. ... (iv)

(3) We have $\tan 2A = \tan (A + A)$

$= \dfrac{\tan A + \tan A}{1 - \tan A \cdot \tan A} = \dfrac{2 \tan A}{1 - \tan^2 A} = \dfrac{2t}{1 - t^2}$, where $t = \tan A$.

(b) Triple angle formulae.

(1) We have $\sin 3A = \sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$= 2 \sin A \cos^2 A + (1 - 2 \sin^2 A) \sin A$

$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A = 3 \sin A - 4 \sin^3 A$.

(2) We have $\cos 3A = \cos (2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A$

$= 2 \cos^3 A - \cos A - 2 (1 - \cos^2 A) \cos A = 4 \cos^3 A - 3 \cos A$.

(3) We have $\tan 3A = \tan (2A + A) = \dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$= \dfrac{\dfrac{2t}{1 - t^2} + t}{1 - \dfrac{2t}{1 - t^2} \cdot t} = \dfrac{3t - t^3}{1 - 3t^2}$, where $t = \tan A$.

(c) Quadruple angle Formulae.

(1) We have $\sin 4A = 2 \sin 2A \cos 2A$

$= 2 (2 \sin A \cos A)(2 \cos^2 A - 1)$

$= 4 \sin A (2 \cos^3 A - \cos A)$. ... ... (i)

Again, $\sin 4A = 4 \sin A \cos A (1 - 2 \sin^2 A)$

$= 4 \cos A (\sin A - 2 \sin^3 A)$. ... (ii)

(2) We have $\cos 4A = 2 \cos^2 2A - 1$

$= 2 (2 \cos^2 A - 1)^2 - 1 = 2 (4 \cos^4 A - 4 \cos^2 A + 1) - 1$

$= 8 \cos^4 A - 8 \cos^2 A + 1$. ... (i)

Again, $\cos 4A = 1 - 2 \sin^2 2A = 1 - 2 (2 \sin A \cos A)^2$

$= 1 - 8 \sin^2 A (1 - \sin^2 A) = 1 - 8 \sin^2 A + 8 \sin^4 A$. ... (ii)

(d) Trigonometrical ratios of $\pi/5$, i. e., of $36°$.

We have $3(36°) = 108° = 180° - 72° = 180° - 2(36°)$.

$\therefore \quad \sin 3\pi/5 = \sin[\pi - (2\pi/5)] = \sin 2\pi/5.$

$\therefore \quad 3\sin \pi/5 - 4\sin^3 \pi/5 = 2\sin \pi/5 \cdot \cos \pi/5.$

$\therefore \quad 3 - 4\sin^2 \pi/5 = 2\cos \pi/5, \quad \because \sin \pi/5 \neq 0.$

$\therefore \quad 3 - 4(1 - \cos^2 \pi/5) = 2\cos \pi/5.$

Writing $\cos \pi/5 = t$, we get $4t^2 - 2t - 1 = 0$.

$$\therefore \quad t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2 \cdot 4}$$

$$= \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4}.$$

But $\cos \pi/5 > 0$, $\therefore \quad \cos \pi/5 = \frac{1+\sqrt{5}}{4}$. ... (i)

Now $\sin^2 \pi/5 = 1 - \cos^2 \pi/5 = 1 - \left(\frac{1+2\sqrt{5}+5}{16}\right) = \frac{10-2\sqrt{5}}{16}$.

$$\therefore \quad \sin \pi/5 = + \frac{\sqrt{10-2\sqrt{5}}}{4}, \quad \because \sin \pi/5 > 0. \qquad \text{... (ii)}$$

From (i) and (ii), we get $\tan \pi/5 = \frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$. ... (iii)

(12) Half angle formulae.

(1) We have $\cos 2\alpha = 1 - 2\sin^2 \alpha$.

$$\therefore \quad 2\sin^2 \alpha = 1 - \cos 2\alpha.$$

$$\therefore \quad \sin \alpha = \pm \sqrt{\frac{1-\cos 2\alpha}{2}}.$$

$$\therefore \text{ we can write } \sin \frac{A}{2} = \pm \sqrt{\frac{1-\cos A}{2}}. \qquad \text{... (i)}$$

(2) We have $\cos 2\alpha = 2\cos^2 \alpha - 1$.

$$\therefore \quad 2\cos^2 \alpha = 1 + \cos 2\alpha.$$

$$\therefore \quad \cos \alpha = \pm \sqrt{\frac{1+\cos 2\alpha}{2}}.$$

$$\text{we have } \cos \frac{A}{2} = \pm \sqrt{\frac{1+\cos A}{2}}. \qquad \text{... (ii)}$$

(3) From (i) and (ii), we have by division

$$\tan \frac{A}{2} = \pm \sqrt{\frac{1-\cos A}{1+\cos A}}. \qquad \ldots \text{(iii)}$$

The ambiguity in sign in each case can be determined if we know the quadrant in which the angle $A/2$ lies.

Illustrative Examples.

Ex. 1. Find (i) sin 15°, cos 15° and tan 15°
and (ii) sin ($22\frac{1}{2}$)°, cos ($22\frac{1}{2}$)° and tan ($22\frac{1}{2}$)°
by using the half angle formulae.

(i) $\sin 15^\circ = \pm\sqrt{\frac{1-\cos 30^\circ}{2}} = +\sqrt{\frac{1-\sqrt{3}/2}{2}}$, $\because 0 < 15^\circ < 90^\circ$

$$= \sqrt{\frac{2-\sqrt{3}}{4}} = \sqrt{\frac{4-2\sqrt{3}}{8}} = \sqrt{\frac{(\sqrt{3}-1)^2}{(2\sqrt{2})^2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}. \qquad \ldots (1)$$

$$\cos 15^\circ = \sqrt{\frac{1+\cos 30^\circ}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \sqrt{\frac{4+2\sqrt{3}}{8}} = \sqrt{\frac{(\sqrt{3}+1)^2}{(2\sqrt{2})^2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}. \qquad \ldots (2)$$

From (1) and (2), we get $\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1}$.

(ii) $\sin (22\frac{1}{2})^\circ = \pm\sqrt{\frac{1-\cos 45^\circ}{2}}$

$$= +\sqrt{\frac{1-1/\sqrt{2}}{2}}, \text{ since } 0 < 22\tfrac{1}{2}^\circ < 90^\circ$$

$$= \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = \tfrac{1}{2}\sqrt{2-\sqrt{2}}. \qquad \ldots (3)$$

Similarly, $\cos (22\frac{1}{2})^\circ = \pm\sqrt{\frac{1+\cos 45^\circ}{2}} = \pm\sqrt{\frac{1+1/\sqrt{2}}{2}}$

$$= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \tfrac{1}{2}\sqrt{2+\sqrt{2}}, \because 0 < 22\tfrac{1}{2}^\circ < 90^\circ. \ldots (4)$$

From (3) and (4) we have, by division

$$\tan (22\tfrac{1}{2})^\circ = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1. \qquad \ldots (5)$$

Note : The trigonometrical ratios of 15° can also be obtained by observing that $15° = 45° - 30°$ and using the subtraction formulae. Since 75° is the complementary angle of 15°, the trigonometrical ratios of 75° can be obtained if the ratios of 15° are known. Similarly the trigonometrical ratios of an angle which can be expressed as a sum (difference), or twice, thrice, or half of an angle, whose ratios are known, can be obtained.

Ex. 2. Show that

$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y.$$

We have $\sin(x+y)\sin(x-y)$

$= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$

$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$

$= \sin^2 x (1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y$

$= \sin^2 x - \sin^2 y =$ the right hand side.

Ex. 3. Show that $\dfrac{\sin 7x - \sin 3x}{\cos 7x + \cos 3x} = \tan 2x.$

We have $\dfrac{\sin 7x - \sin 3x}{\cos 7x + \cos 3x}$

$$= \frac{2\cos\left(\frac{7x+3x}{2}\right)\sin\left(\frac{7x-3x}{2}\right)}{2\cos\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)}$$

$$= \frac{2\cos 5x \sin 2x}{2\cos 5x \cos 2x}$$

$= \tan 2x$, if $\cos 5x \neq 0$ and $\cos 2x \neq 0$.

Ex. 4. Show that

$$\frac{\sin A + \sin 5A + \sin 9A}{\cos A + \cos 5A + \cos 9A} = \tan 5A.$$

When is this result not valid ?

Left hand side $= \dfrac{\sin A + \sin 9A + \sin 5A}{\cos A + \cos 9A + \cos 5A}$

$$= \frac{2\sin\left(\frac{A+9A}{2}\right)\cos\left(\frac{9A-A}{2}\right) + \sin 5A}{2\cos\left(\frac{9A+A}{2}\right)\cos\left(\frac{9A-A}{2}\right) + \cos 5A}$$

$$= \frac{2\sin 5A \cos 4A + \sin 5A}{2\cos 5A \cos 4A + \cos 5A}$$

$$= \frac{\sin 5A\,(2\cos 4A + 1)}{\cos 5A\,(2\cos 4A + 1)}$$

$$= \frac{\sin 5A}{\cos 5A}, \quad \text{if } \cos 4A \neq -1/2,\ \cos 5A \neq 0,$$

$= \tan 5A =$ the right-hand side.

Ex. 5. Find the values of $\sin 2\alpha$ and $\cos 2\alpha$ if $13 \sin \alpha = 5$ and $\pi/2 < \alpha < \pi$.

We have $\sin \alpha = 5/13$.

$\therefore \quad \cos^2 \alpha = 1 - \sin^2 \alpha = 1 - 25/169 = 144/169.$

$\therefore \quad \cos \alpha = -12/13$, ($-$ ve, as α is an obtuse angle).

$$\therefore \quad \sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \cdot \frac{5}{13}\left(\frac{-12}{13}\right) = -\frac{120}{169},$$

$$\text{and} \quad \cos 2\alpha = 1 - 2\sin^2 \alpha = 1 - 2 \cdot \frac{25}{169} = \frac{119}{169}.$$

Ex. 6. prove that

$$\sin^2 A + \sin^2 (120^\circ + A) + \sin^2 (120^\circ - A) = 3/2.$$

We have $2 \sin^2 x = 1 - \cos 2x$.

Hence the expression on the left hand side

$$= \tfrac{1}{2}(1 - \cos 2A) + \tfrac{1}{2}[1 - \cos (240^\circ + 2A)]$$
$$+ \tfrac{1}{2}[1 - \cos (240^\circ - 2A)]$$
$$= \tfrac{1}{2}[3 - \cos 2A - \cos (240^\circ + 2A) - \cos (240^\circ - 2A)]$$
$$= \tfrac{1}{2}[3 - \cos 2A - 2 \cos 240^\circ \cos 2A]$$
$$= \tfrac{1}{2}[3 - \cos 2A + \cos 2A], \quad \because \quad \cos 240^\circ = -1/2,$$
$$= 3/2 = \text{the right-hand side.}$$

EXERCISE 6 (b)

(1) Find the values of $\cos (x - y)$, $\sin (x + y)$ and $\tan (x+y)$ if $\sin x = 3/5$, $\sin y = 12/13$, where x is an acute angle and y is an obtuse angle.

(2) If $\tan x = 3/4$ and $\tan y = 1/7$, show that $\tan (x + y) = 1$.
State the value of $(x + y)$ if x and y are acute.

(3) Show that

(i) $\sin x - \cos x = \sqrt{2} \sin (x - \pi/4)$,

(ii) $\sqrt{3} \cos x + \sin x = 2 \cos (x - \pi/6)$,

(iii) $\cos 20^\circ - \sin 20^\circ = \sqrt{2} \sin 25^\circ$.

(4) Show that

(i) $$\frac{\sin x + \sin y}{\sin x - \sin y} = \tan\left(\frac{x+y}{2}\right) \cot\left(\frac{x-y}{2}\right),$$

(ii) $$\frac{\cos x + \cos y}{\cos x - \cos y} = \cot\left(\frac{x+y}{2}\right) \cot\left(\frac{y-x}{2}\right).$$

(5) Prove that

(i) $$\frac{\sin x - \sin y}{\cos x + \cos y} = \tan\left(\frac{x-y}{2}\right);$$

(ii) $$\frac{\sin x + \sin y}{\cos x - \cos y} = \cot\left(\frac{y-x}{2}\right).$$

(6) Show that

(i) $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$,

(ii) $\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$.

(7) Prove that

(i) $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$,

(ii) $\cos 3x - \cos 5x + \cos 7x - \cos 9x = 4\sin x \cos 2x \sin 6x$.

(8) Prove that

(i) $\dfrac{\sin x + \sin 5x + \sin 9x}{\cos x + \cos 5x + \cos 9x} = \tan 5x$.

(ii) $\dfrac{\sin x + \sin 2x + \sin 3x + \sin 4x}{\cos x + \cos 2x - \cos 3x - \cos 4x} = \cot x$.

(9) Show that

(i) $\cos 15^\circ + \cos 45^\circ + \cos 75^\circ + \cos 105^\circ = \sqrt{3}\cos 15^\circ$,

(ii) $\cos 40^\circ + \cos 80^\circ - \cos 10^\circ - \cos 130^\circ = \sqrt{2}\sin 25^\circ$.

(10) Prove that

(i) $\tan 80^\circ - \tan 35^\circ = 1 + \tan 80^\circ \tan 35^\circ$,

(ii) $\tan 20^\circ \tan 30^\circ + \tan 30^\circ \tan 40^\circ + \tan 40^\circ \tan 20^\circ = 1$.

(11) Prove that $\sin 8x = 8 \sin x \cos x \cos 2x \cos 4x$.

(12) Prove that

(i) $\sin 3x = 4 \sin x \sin(120^\circ + x)\sin(120^\circ - x)$,

(ii) $\cos 3x = 4 \cos x \cos(120^\circ + x)\cos(120^\circ - x)$.

(13) Prove that

(i) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = 3/16$,

(ii) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1/16$.

(14) Find the values of $\sin 2x$ and $\cos 2x$ if

(i) $\cos x = 3/5$, (x is an acute angle)

(ii) $\sin x = 12/13$, (x is an obtuse angle)

(iii) $\tan x = -15/8$.

Prove that Exs. (15 to 20)

(15) (i) $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$, (ii) $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$.

(iii) $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x$.

(16) (i) $\dfrac{\cos 2x}{1 - \sin 2x} = \tan\left(\dfrac{\pi}{4} + x\right)$,

(ii) $\dfrac{\cos 2x}{1 + \sin 2x} = \tan\left(\dfrac{\pi}{4} - x\right)$.

(17) (i) $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4\cos^2[(x-y)/2]$.

(ii) $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2[(x-y)/2]$.

(18) $\tan x \tan 2x \tan 3x = \tan 3x - \tan 2x - \tan x$.

(19) $\sin(y-z)\cos x + \sin(z-x)\cos y + \sin(x-y)\cos z = 0$.

(20) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$, if $A + B + C = 180^\circ$.

(21) Fill in the blanks in the following :

(i) (*a*) $\sin 150^\circ - \cos\theta = 0, 0 < m(\theta) < \frac{\pi}{2}$. $\therefore m(\theta) =$ ..... ;

(*b*) $\cos(105^\circ) =$ .....

(*c*) $\sin 54^\circ \cos 36^\circ + \cos 54^\circ \sin 36^\circ =$ .....

(*d*) $\tan(-765^\circ) =$ .....

(*e*) $\tan(-570^\circ) =$ .....

(ii) $\cos 25^\circ + \cos 50^\circ + \sin 30^\circ + \cos 205^\circ + \cos 130^\circ =$ .....

(iii) $\tan\frac{\pi}{10}\tan\frac{2\pi}{10}\tan\frac{3\pi}{10}\tan\frac{4\pi}{10} =$ .....

(iv) $\cos(x+y)\cos(x-y) = \cos^2 x -$ .....

(v) $\tan 20^\circ + \tan 25^\circ + \tan 20^\circ \tan 25^\circ =$ .....

(vi) $\sin\frac{5\pi}{12} - \cos\frac{5\pi}{12} =$ .....

(vii) $\cos 10^\circ + \cos 20^\circ + \cos 70^\circ + \cos 80^\circ = \sqrt{6}\cos$ .....

(viii) If $\theta = 440^\circ$, the sign to be taken in the formula

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$ is .....

(ix) $8\sin\theta\cos\theta\cos 2\theta\cos 4\theta = \sin$ .....

(x) If $\cos\theta = \frac{3}{5}$, then $\sin(\pi-\theta)\cos\left(\frac{\pi}{2}+\theta\right) =$ .....

(22) Choose in each case the correct answer from (*a*), (*b*), (*c*), (*d*), given below :

(i) If $5\sin\theta = 4$ and $\frac{\pi}{2} < \theta < \pi$ then $\sec\theta =$ .....

(*a*) $-\frac{3}{5}$, (*b*) $\frac{5}{3}$, (*c*) $-\frac{5}{3}$ (*d*) none of these.

(ii) $\sin^2 60^\circ \cos^2 30^\circ \cot 45^\circ \cos^2 70^\circ =$ .....

(*a*) 1, (*b*) $\frac{1}{16}$, (*c*) $\frac{1}{8}$, (*d*) none of these.

(iii) $\cos 2\theta + \cos 4\theta =$ .....

(*a*) $2\sin\theta\sin 3\theta$, (*b*) $2\cos\theta\cos 3\theta$, (*c*) $2\cos(-\theta)\sin 3\theta$.

(iv) $\sin 120^\circ - \cos 30^\circ =$ .....

(*a*) $2\cos^2 60^\circ$, (*b*) $2\sin 90^\circ \sin 30^\circ$, (*c*) 0, (*d*) $2\sin^2 60^\circ$.

(v) $\frac{\operatorname{cosec}\theta}{\cos\theta} + \frac{\sec\theta}{\sin\theta} =$ .....

(*a*) $2\operatorname{cosec} 2\theta$, (*b*) $2\cot 2\theta$, (*c*) $2\sec 2\theta$, (*d*) $4\operatorname{cosec} 2\theta$.

ANSWERS

(1) 16/65, 33/65, $-33/56$ (2) $\pi/4$. (14) (i) 24/25; $-7/25$.
(ii) $-120/169$; $-119/169$. (iii) $-240/289$; $-161/289$.

(21) (i) (a) $\pi/3$, (b) $-(\sqrt{3}-1)/2\sqrt{2}$, (c) 1, (d) -1, (e) $-1/\sqrt{3}$.
(ii) 1/2, (iii) 1, (iv) $\sin^2 y$, (v) 1, (vi) $1/\sqrt{2}$, (vii) 5°, (viii) – ve,
(ix) 8θ, (x) $-16/25$.

(22) (i) (c), (ii) (d), (iii) (b), (iv) (c), (v) (d).

□ □ □

CHAPTER 7

Properties of a Triangle

(1) **Introduction.** Let ABC be any triangle. The measures of its angles at its vertices A, B, C will be denoted by A, B, C respectively and the lengths of its sides opposite to them will be denoted by a, b, c respectively. Every triangle has these six elements. These six elements are related to one another in a special way. We propose to consider first the Sine rule and the Cosine rule which contain important relationships.

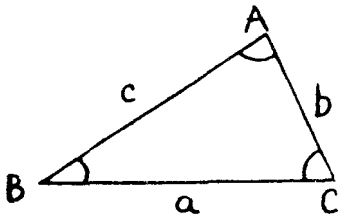


Fig 7·1

(2) **The Sine rule. In any $\triangle ABC$, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.**

Let ABC be any triangle.

(i) Choosing B as the origin and BxC and By as the axes of co-ordinates, the co-ordinates of the point A are $(c \cos B, c \sin B)$.

(ii) Now shifting the origin to the point C and taking BCX and CY as the axes of co-ordinates, the co-ordinates of A are

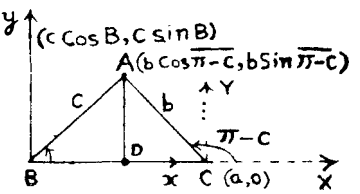


Fig. 7·2

$$(b \cos \overline{\pi - C},\ b \sin \overline{\pi - C}),\ \text{i. e. } (-b \cos C,\ b \sin C).$$

Now use the transformation formula for the ordinate, viz. $y = k + Y$, for the point A. Here $y = c \sin B$, $k = 0$, $Y = b \sin (\pi - C)$ i. e. $b \sin C$.

Substituting in the transformation formula for the ordinate,

we get $c \sin B = b \sin C$, i. e., $\frac{b}{\sin B} = \frac{c}{\sin C}$.

Similarly, by taking C and A as the old and new origins and CA as the common x-axis we shall get $\frac{c}{\sin C} = \frac{a}{\sin A}$.

$\therefore$ we have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

(3) **In any $\triangle ABC$, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circum-radius of $\triangle ABC$.**

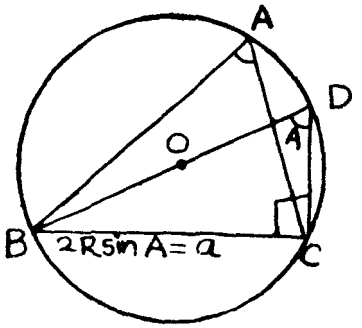


Fig. 7·3 (i)

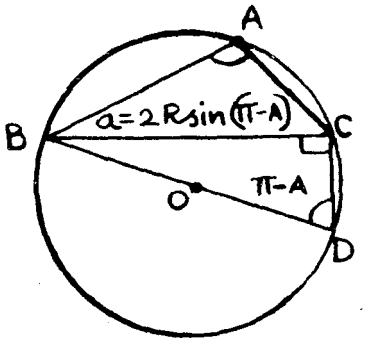


Fig. 7·3 (ii)

In Fig. 7·3 (i), $\angle A$ is acute. Let seg BD ($=2R$, in length) be a diameter of the circumcircle of the $\triangle ABC$.

Then $BC = a$ and from the right angled $\triangle BCD$, we get $a = 2R \sin A$. Similarly, $b = 2R \sin B$ and $c = 2R \sin C$.

In Fig. 7·3 (ii), $\angle A$ is obtuse and $a = BD \cdot \sin(\pi - A) = 2R \sin A$.

If $\angle A = \pi/2$ then $a = 2R = 2R \sin A$, $\because \sin A = 1$.

Thus for any measure of $\angle A$, we have

$a = 2R \sin A$ and similarly $b = 2R \sin B$, $c = 2R \sin C$.

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

The result obtained here may be used to deduce the sine-rule.

(4) **The Cosine rule. In any $\triangle ABC$, $a^2 = b^2 + c^2 - 2bc \cos A$.**

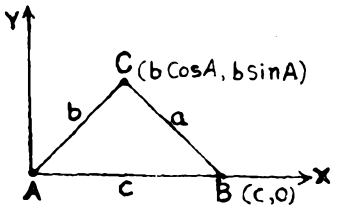


Fig. 7·4

Taking A as the origin and ABX and AY as the axes of co-ordinates, we have $B \equiv (c, 0)$ and $C \equiv (b \cos A, b \sin A)$.

$\therefore a^2 = BC^2 = (b \cos A - c)^2 + (b \sin A - 0)^2$. See Fig. 7·4.

$\therefore a^2 = b^2 (\cos^2 A + \sin^2 A) - 2bc \cos A + c^2$.

$\therefore a^2 = b^2 + c^2 - 2bc \cos A$. ... (i)

Similarly choosing B as the origin and BC as the x-axis, we get

$$b^2 = c^2 + a^2 - 2\,c\,a\cos B, \qquad \ldots (ii)$$

and choosing C as the origin and CA as the x-axis, we get

$$c^2 = a^2 + b^2 - 2\,ab\cos C. \qquad \ldots (iii)$$

Results (i), (ii) and (iii) constitute the Cosine rule for a triangle.

(5) **Projection rule. In any $\triangle ABC$, $a = b\cos C + c\cos B$.**

In Fig 7·2, $A \equiv (c\cos B, c\sin B)$ with reference to the axes Bx, By. Let the origin be shifted to the point C whose co-ordinates with reference to the old axes are $(a, 0)$.

Now use the transformation formula for the abscissa viz $x = h + X$ for the point A. We have $x = c\cos B$, $h = a$, $X = b\cos(\pi - C)$ i.e. $-b\cos C$.

$\therefore$ we get $c\cos B = a - b\cos C$.

$$\therefore\ a = b\cos C + c\cos B. \qquad \ldots (i)$$

Similarly $\quad b = c\cos A + a\cos C, \qquad \ldots (ii)$

and $\quad c = a\cos B + b\cos A. \qquad \ldots (iii)$

Alternative method for deducing the Cosine Rule.

Hint. Multiply (i), (ii), (iii), by a, b, c respectively and subtract one of the results so obtained from the sum of the remaining two.

(6) **Napier's Formula : The Tangent Law : In any triangle ABC,**

$$\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)} = \frac{a-b}{a+b}.$$

We have $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$, i. e., $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$.

$$\therefore\ \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a-b}{a+b}.$$

$$\therefore\ \frac{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} = \frac{a-b}{a+b}.$$

Thus, $\dfrac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)} = \dfrac{a-b}{a+b}$.

The result can also be written as

$$\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\tan\left(\frac{A+B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right).$$

(7) Expressions for (i) sin $(A/2)$, (ii) cos $(A/2)$, (iii) tan $(A/2)$ in terms of the sides of a triangle ABC.

(i) We use the formula $2\sin^2(A/2) = 1 - \cos A$.

$$\therefore\ 2\sin^2(A/2) = 1 - \frac{b^2+c^2-a^2}{2bc}, \text{ on using the cosine rule,}$$

$$= \frac{2bc-(b^2+c^2-a^2)}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc}.$$

Now let $2s = a+b+c$ so that $a+b = 2s-c$.

$\therefore\ a+b-c = 2(s-c)$ and similarly $a-b+c = 2(s-b)$.

$$\therefore\ 2\sin^2(A/2) = \frac{2(s-c)\cdot 2(s-b)}{2bc}.$$

$$\therefore\ \sin^2(A/2) = \frac{(s-b)(s-c)}{bc}.$$

$$\therefore\ \sin(A/2) = +\sqrt{\frac{(s-b)(s-c)}{bc}},$$ + ve sign being taken as $\angle(A/2)$ is acute. (i)

(ii) We use the formula $2\cos^2(A/2) = 1 + \cos A$.

$$\therefore\ 2\cos^2(A/2) = 1 + \frac{b^2+c^2-a^2}{2bc}, \text{ on using the cosine rule,}$$

$$= \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{2s\cdot 2(s-a)}{2bc}.$$

$$\therefore\ \cos^2(A/2) = \frac{s(s-a)}{bc}.$$

$\therefore \cos(A/2) = +\sqrt{\frac{s(s-a)}{bc}}$, + sign being taken as $\angle(A/2)$ is acute. ... ... (ii)

(iii) We have $\tan(A/2) = \frac{\sin(A/2)}{\cos(A/2)}$.

$\therefore \tan(A/2) = \frac{\sqrt{(s-b)(s-c)/bc}}{\sqrt{s(s-a)/bc}}$, on using (i) and (ii),

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

(8) The area of a triangle. If $\triangle$ be the area of any triangle ABC, then

(i) $\triangle = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$.

(ii) $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$, **where** $2s = a+b+c$.

(i) The area of a triangle $= \frac{1}{2} \times$ base $\times$ height.

$\therefore \triangle = \frac{1}{2} BC \cdot AD$, on using Fig. 7·2

$= \frac{1}{2} ac \sin B$, $\because AD = c \sin B$.

Similarly, $\triangle = \frac{1}{2} bc \sin A$ and $\triangle = \frac{1}{2} ca \sin B$.

(ii) We use the formula : $\triangle = \frac{1}{2} bc \sin A$.

$\therefore \triangle = \frac{1}{2} bc \cdot 2 \sin(A/2) \cdot \cos(A/2)$.

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}},$$

on using (i) and (ii) of § 7.

$\therefore \triangle = \sqrt{s(s-a)(s-b)(s-c)}$.

This result gives the area of a triangle in terms of its sides.

(9) Expressions for (i) circum-radius R and (ii) in-radius r of a triangle.

(i) We use the formula : $\triangle = \frac{1}{2} bc \sin A$.

$\therefore \triangle = \frac{1}{2} bc \frac{a}{2R}$, $\because \frac{a}{\sin A} = 2R$, (refer to § 3.)

$$= \frac{abc}{4R}.$$

$$\therefore R = \frac{abc}{4\triangle}.$$

(ii) We use the result :

Area of ΔABC = Area of ΔIBC + Area of ΔICA + Area of ΔIAB,

where I is the centre of the inscribed circle.

$$\therefore \quad \Delta = \tfrac{1}{2} ar + \tfrac{1}{2} br + \tfrac{1}{2} cr$$

$$= \frac{r}{2}(a+b+c) = rs, \quad \because \ a+b+c = 2s.$$

$$\therefore \quad r = \frac{\Delta}{s}.$$

Illustrative Examples

Ex. 1. In any ΔABC, prove that

$$a \sin(B-C) + b \sin(C-A) + c \sin(A-B) = 0.$$

Using the result $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, we have

$$\text{L. H. S.} = 2R \sin A \sin(B-C) + 2R \sin B \sin(C-A) + 2R \sin C \sin(A-B),$$

$$= R[2 \sin(B+C) \sin(B-C) + 2 \sin(C+A) \sin(C-A) + 2 \sin(A+B) \sin(A-B)], \quad \because \ A+B+C = \pi,$$

$$= R[(\cos 2C - \cos 2B) + (\cos 2A - \cos 2C) + (\cos 2B - \cos 2A)],$$

$$= R[0] = 0 = \text{R. H. S.}$$

Ex. 2. In any ΔABC, prove that

$$\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}.$$

Using the result

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, = k, \text{ say, we have}$$

$$\text{L. H. S.} = \frac{k(\sin A + \sin B)}{k \sin C}$$

$$= \frac{2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)}{2 \sin \frac{C}{2} \cos \frac{C}{2}}$$

$$= \frac{\cos\left(\frac{C}{2}\right) \cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right) \cos\left(\frac{C}{2}\right)}$$

$$\left[\because \sin\left(\frac{A+B}{2}\right) = \sin\left(90 - \frac{C}{2}\right) = \cos\frac{C}{2} \right]$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}, \quad \because \sin\left(\frac{C}{2}\right) = \cos\left(\frac{A+B}{2}\right).$$

$$= \text{R. H. S.}$$

Ex. 3. In any ΔABC, prove that

$$\frac{(b^2 - c^2)\cos A}{a} + \frac{(c^2 - a^2)\cos B}{b} + \frac{(a^2 - b^2)\cos C}{c} = 0.$$

On using the cosine rule, we have

$$\text{L. H. S.} = \frac{(b^2 - c^2)(b^2 + c^2 - a^2)}{2abc} + \frac{(c^2 - a^2)(c^2 + a^2 - b^2)}{2abc}$$

$$+ \frac{(a^2 - b^2)(a^2 + b^2 - c^2)}{2abc}$$

$$= \frac{1}{2abc}[\{(b^2 - c^2)(b^2 + c^2) + (c^2 - a^2)(c^2 + a^2)$$

$$+ (a^2 - b^2)(a^2 + b^2)\}$$

$$- \{a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2)\}]$$

$$= \frac{1}{2abc}[\{(b^4 - c^4) + (c^4 - a^4) + (a^4 - b^4)\} - \{\Sigma a^2(b^2 - c^2)\}]$$

$$= \frac{1}{2abc}[\{0\} - \{0\}] = 0 = \text{R. H. S.}$$

Ex. 4. In any ΔABC, prove that

$$\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ac} = \frac{\cos C}{c} + \frac{c}{ab}.$$

Using the cosine rule, we have

$$\frac{\cos A}{a} + \frac{a}{bc} = \frac{b^2 + c^2 - a^2}{2abc} + \frac{a}{bc} = \frac{b^2 + c^2 - a^2 + 2a^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc}.$$

Similarly, $\frac{\cos B}{b} + \frac{b}{ac} = \frac{a^2 + b^2 + c^2}{2abc} = \frac{\cos C}{c} + \frac{c}{ab}$.

Hence the result follows.

Ex. 5. If Δ is the area of ΔABC, prove that

$$\Delta = \frac{a^2 - b^2}{2} \cdot \frac{\sin A \sin B}{\sin(A - B)}.$$

Using the result $a = 2R \sin A$, $b = 2R \sin B$, we have

$$\frac{a^2 - b^2}{2} \cdot \frac{\sin A \sin B}{\sin(A - B)} = \frac{4R^2(\sin^2 A - \sin^2 B)\sin A \sin B}{2\sin(A - B)}$$

$= 2R^2 \sin(A+B) \sin A \sin B$,

$[\because \sin^2 A - \sin^2 B \equiv \sin(A+B)\sin(A-B)]$

$= 2R^2 \sin C \sin A \sin B$, $\because \sin(A+B) = \sin C$,

$= (2R \sin A) \cdot (2R \sin B) \cdot \frac{1}{2} \sin C$

$= \frac{1}{2} ab \sin C = \Delta$, $\because a = 2R \sin A$, $b = 2R \sin B$.

Ex. 6. In a ΔABC, if $m \angle A = 60°$, $b = 1 + \sqrt{3}$, $c = 2$, find a, $m(\angle B)$ and $m(\angle C)$.

By the cosine rule, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\therefore \frac{1}{2} = \frac{(1+\sqrt{3})^2 + 4 - a^2}{2(1+\sqrt{3}) \cdot 2}, \text{ on using the given data.}$$

$$\therefore 2(1+\sqrt{3}) = 1 + 3 + 2\sqrt{3} + 4 - a^2.$$

$$\therefore a^2 = 6, \quad \therefore a = \sqrt{6}. \qquad \ldots (1)$$

Now using the result, $\frac{a}{\sin A} = \frac{c}{\sin C}$,

we have $\sin C = \frac{c \sin A}{a} = 2\left(\frac{\sqrt{3}}{2}\right) \Big/ \sqrt{6} = \frac{1}{\sqrt{2}}$.

$\therefore m(\angle C) = 45°$ or $135°$.

But $135°$ is inadmissible, since $m(\angle A + \angle C) = 195° > 180°$.

$\therefore m(\angle C) = 45°$ and hence $m(\angle B) = 75°$.

EXERCISE 7 (*a*)

(1) Fill in the blanks in the following :

(i) In a triangle ABC, $a = 8$ and $m(\widehat{BAC}) = 60°$.
$\therefore$ its circum-radius = ...

(ii) In a traingle ABC, $a = 5$, $m(\angle A) = 45°$, $m(\angle B) = 60°$;
$\therefore$ b = ..., c = ..., area (ΔABC) = ...

(iii) In a traingle ABC, $a = 49$, $\cos A = \frac{24}{25}$;
$\therefore$ its circum-radius = ...

(iv) In a triangle ABC, $a = 3$, $b = 4$, $c = 5$;
$\therefore$ its circum-radius = ... and its in-radius = ...

(v) In a traingle ABC, $\cos A = \frac{12}{13}$, $\sin B = \frac{3}{5}$ and the circum-radius $R = 26$.
$\therefore$ a = ... and b = ...

(vi) The sides of a triangle ABC are 10, 12 and 18 units.
$\therefore$ area of the triangle ABC = ... sq. units and its in-radius = ... units.

(2) State whether the following statement is true or false. "If $a = 10$ cm., $c = 8$ cm. and $m(\angle B) = 30^\circ$, then the area of $\triangle ABC$ changes if a and c remain unchanged but $m(\angle B)$ is increased to 150°".

(3) In a $\triangle ABC$, $m\angle B = 30^\circ$, $c = 1.5$ m. and $b = 50\sqrt{3}$ cms. Determine the measures of the other angles and the length of the third side.

(4) In a $\triangle ABC$, $\cos A = 17/22$, and $\cos C = 1/14$. Find the ratios $a : b : c$.

(5) The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Find the measure of its greatest angle.

In any triangle ABC (in examples 6 to 10) prove that

(6) (i) $(b-c)\sin A+(c-a)\sin B+(a-b)\sin C=0$,

(ii) $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)+c^2(\cos^2 A-\cos^2 B)=0$,

(iii) $(b-c)^2\cos^2(A/2)+(b+c)^2\sin^2(A/2)=a^2$,

(iv) $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$,

(v) $a\cos A+b\cos B+c\cos C=8\Delta^2/abc$, where Δ = area ($\triangle ABC$).

(vi) $c^2(\tan A-\tan B)=(a^2-b^2)(\tan A+\tan B)$,

(vii) $\left(\frac{b+c}{b-c}\right)\tan(A/2)-\left[\frac{b-c}{b+c}\right]\cot(A/2)=2\cot(B-C)$,

(viii) $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}=0$,

(ix) $b^2\sin 2C+c^2\sin 2B=2bc\sin A$,

(x) $a(b\cos C-c\cos B)=b^2-c^2$,

(xi) $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$,

(xii) $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$,

(xiii) $\sin^2(A/2)+\sin^2(B/2)+\sin^2(C/2)+2\sin(A/2)\sin(B/2)\sin(C/2)=1$,

(xiv) $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$ where Δ = area ($\triangle ABC$),

(xv) $\frac{b\sec B+c\sec C}{\tan B+\tan C}=\frac{abc}{2\Delta}$, where Δ = area ($\triangle ABC$).

(7) If $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$, then show that $m(\angle C)=\pi/3$.

(8) If $\frac{\cos A}{a}=\frac{\cos B}{b}$ then show that $\triangle ABC$ is isosceles.

(9) If $a\cos^2(C/2)+c\cos^2(A/2)=3b/2$ then show that the lengths of the sides of the triangle are in A. P.

(10) If a, b, c are in A. P. show that $3\tan A/2\tan C/2=1$.

(11) Prove that

$$\sin(B-C)+\sin(C-A)+\sin(A-B)=-4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$$

(12) If $a + b + c = 2s$ and Δ is the area of ΔABC, prove that $abc(\sin A + \sin B + \sin C) = 4s\Delta$.

(13) In a ΔABC, prove that $bc\cos^2(A/2) + ca\cos^2(B/2) + ab\cos^2(C/2) = s^2$, where $2s = a + b + c$.

(14) If R, r and Δ denote the circum-radius, the in-radius, and the area of ΔABC, prove that $Rr(\sin A + \sin B + \sin C) = \Delta$.

ANSWERS

(1) (i) $\frac{8\sqrt{3}}{3}$, (ii) $\frac{5\sqrt{6}}{2}, \frac{5(1+\sqrt{3})}{2}, \frac{25(3+\sqrt{3})}{8}$;

(iii) 87·5; (iv) 2·5, 1; (v) 20, 31·2; (vi) $40\sqrt{2}$, $2\sqrt{2}$;

(2) False as $\frac{1}{2}ac\sin 30° = \frac{1}{2}ac\sin 150°$.

(3) $m(\angle A) = 90°$, $m(\angle C) = 60°$, $a = 100\sqrt{3}$ cms. (4) 7 : 9 : 11.

(5) $2\pi/3$.

(10) Use of Trigonometry in problems on heights and distances.

We now illustrate the use of trigonometrical ratios in estimating the distance between two inaccessible points and the heights or depths of points too high or too low to determine by direct measurement. Before we do so we explain the two terms :

Angle of Elevation and Angle of Depression.

It is common experience that we have to raise our eyes to look at objects that are above the level of the eyes and to lower our eyes to look at objects below the level of our eyes. Thus if O is the observer and P is an object which is at a different level from that of O then,

(i) **the angle of elevation** *is the angle XOP determined by the horizontal ray OX, and the ray of line of sight OP, when the object P is at a higher level,* (See Fig. 7·5).

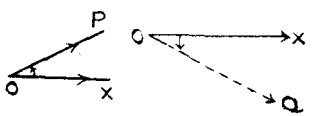


Fig 7·5 Fig 7·6

(ii) **the angle of depression** *is the angle XOQ determined by the horizontal ray OX and the ray of line of sight OQ when the object Q is at a lower level.* (See Fig. 7·6).

Note 1 : We always treat trees, towers or hills as ideal vertical objects which are represented by segments perpendicular to the ground.

Note 2 : We shall always regard the observer or his eye to be in the plane of the ground unless there is specific mention about the observer's height.

Illustrative Examples.

Ex. 1. From the top of a building 30 m. high, the measure of the angle of elevation of the top of a tower was 30° and from the bottom of the building the measure of the angle of elevation was found to be 60°. Find the height of the tower.

In the Fig. 7·7, seg AB represents the building and seg CD represents the tower and seg $BE \perp$ seg CD. Further

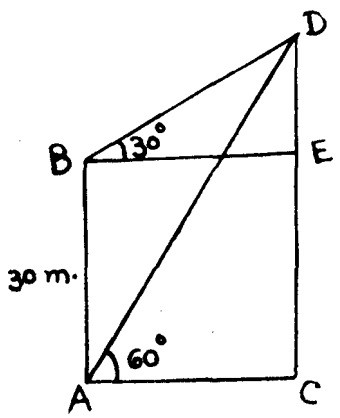


Fig. 7·7

$m(\angle EBD) = 30°$ and

$m(\angle CAD) = 60°$.

In the right-angled $\triangle CAD$,

$CD = AC \tan 60°$

and in the right-angled $\triangle EBD$,

$ED = BE \tan 30°$

$= AC \tan 30°$.

Now $AB = CE$

$= CD - ED$.

$\therefore 30 = AC(\tan 60° - \tan 30°)$.

$\therefore AC = \dfrac{30}{\sqrt{3} - 1/\sqrt{3}} = \dfrac{30\sqrt{3}}{2}$

$= 15\sqrt{3}$ m.

Now $CD = AC \tan 60°$

$= 15\sqrt{3}(\sqrt{3}) = 45$ m.

Ex. 2. From the top of a tower, 50 m. high, the measures of the angles of depression of two objects due north of the tower are 60° and 45°. Find the distance between the two objects.

Let CD be the tower and A and B be the two objects so that CAB is due north of the tower.

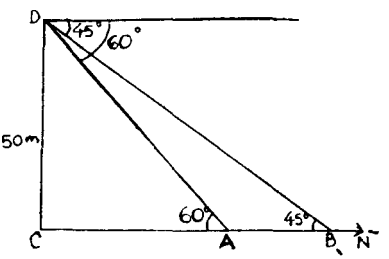


Fig. 7·8

In the right-angled $\triangle CBD$,

$m(\angle CBD) = 45°$.

$\therefore BC = CD = 50$ m:

In the right-angled $\triangle CAD$,

$m(\angle CAD) = 60°$.

$\therefore AC = CD \cot 60° = \dfrac{50}{\sqrt{3}}$.

$\therefore BA = 50 - \dfrac{50}{\sqrt{3}} = \dfrac{50}{\sqrt{3}}(\sqrt{3} - 1)$ m.

$\therefore$ the distance between the two objects is $\dfrac{50(\sqrt{3} - 1)}{\sqrt{3}}$ m.

Ex. 3. The measure of the angle of elevation of a cloud from a point 50 m. above a lake is 30° and the measure of the angle of depression of its reflection in the lake is 60°. Find the height of the cloud.

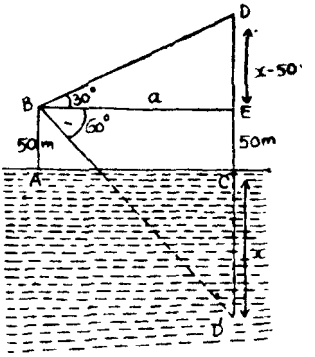


Fig. 7·9

Let seg AC represent the surface of the lake and D and D' denote the cloud and its reflection in the lake so that seg AC bisects seg $D'D$ at right angles.

Let $CD = x$;

then $D'E = D'C + CE = CD + CE = x + 50$,

and $ED = CD - CE = x - 50$.

From the right-angled $\triangle BED$, we have

$ED = BE \tan 30°$.

From the right-angled $\triangle BED'$,

we have $ED' = BE \tan 60°$.

$$\therefore \frac{ED'}{ED} = \frac{\tan 60°}{\tan 30°} = 3. \quad \therefore \frac{x + 50}{x - 50} = 3. \quad \therefore \quad x = 100 \text{ m.}$$

EXERCISE 7 (b)

(1) The measure of the angle of elevation of the top P of a tree PQ from a point A is 30°; and from a point B where $AB = a$ and ABQ is horizontal, the measure of the angle of elevation is 60°. Prove that

$$PQ = \frac{a\sqrt{3}}{2}.$$

(2) A pillar subtends at a point, d metres apart from its foot, the same angle as that subtended at the same point by a statue on its top. If the pillar is h m. high show that the height of the statue is

$$\frac{h(d^2 + h^2)}{d^2 - h^2} \text{ metres.}$$

(3) From the deck of a ship the elevation of the top of a cliff is 50°; from the top of the mast 20 m. high the elevation is 35°. Find the height of the cliff to the nearest metre. Use Trigonometrical Tables.

(4) The measure of the angle of elevation of the top of a pole is 15° from a point on the ground. On walking 50 m. towards the pole the measure of the angle of elevation is found to be 30°. Find the height of the pole.

(5) The roof of a house is h m. above the lake. The measure of the angle of elevation of a balloon from the roof is α and the measure of the angle of depression of the reflection of the balloon in the lake is 2α. Show that the height x of the balloon above the lake is given by $x = \frac{h(3 - \tan^2 \alpha)}{\sec^2 \alpha}$.

(6) A bronze statue of height h m. is mounted on a marble pedestal. The measures of the angles of elevation of the top and bottom of the statue from a height x m. above the ground level are α and β. Prove that the height of the pedestal is

$$\frac{(h - x)\tan\beta + x\tan\alpha}{\tan\alpha - \tan\beta}.$$

(7) A man walking due north observes that the measure of the angle of elevation of a balloon, which is due East of him and is flying towards north-west, is 60°. After walking 450 m. the balloon is directly over his head. Find the height of the balloon correct to the metre assuming the balloon to be flying at the same height.

(8) From a helicopter the measures of the angles of depression of the top and bottom of a tower, the height of which is h m., are observed to be α and β respectively. Show that the horizontal distance of the helicopter from the tower is $h\cos\alpha\cos\beta\sin(\beta - \alpha)$ m.

(9) The top of a tower has an elevation of 40° from a point on one side and 50° from a point on the other side directly opposite to the first point. If the two points are 150 m. apart, find the height of the tower, using trigonometric tables.

(10) A flag-staff of unknown height stands on the edge of a cliff. From a point 500 m. from the base of the cliff an observer measures the angle of elevation of the top and bottom of the flag-staff and finds them to be 17° and 14° 32′ respectively. Find the height of the flag-staff by using trigonometric tables.

(11) Fill in the blanks in the following :

(i) The angle of elevation of the top of a tree of height $20\sqrt{3}$ m. is 60°. Hence the distance of the tree from the observer is ..... m.

(ii) If the angle of elevation of a point A from a point B is 37°, then the angle of depression of B from A is ..... degrees.

ANSWERS

(3) 48·4 m. (4) 25 m. (7) 779 m. (9) 73·86 m. (10) 23·25 m.
(11) (i) 20, (ii) 37.

□ □ □

CHAPTER 8

Vectors

(1) **Directed Segments** : Suppose a fielder on a cricket field is asked to move 10 meters from where he is. Is this enough to tell him where he should go ? All points on the circumference of a circle with his present position, say A, as centre and a radius of 10 meters will be at a distance of 10 meters from A, and the fielder knows only that his new position B will be one of these points. However, if he is further told that he should move towards the ' leg ' side, he will know where to go. Thus, to **specify the ' displacement '** of a body, we must give both the **direction** of the displacement and the **distance** through which the body moves. A displacement, like the one above, can be represented geometrically by a **directed segment** as defined below.

Directed Line Segment. Definition : *A portion of a given straight line between any two points on it is called a* ***directed segment****, provided one of the two points is distinguished as the initial point and the other as the terminal point. The direction of the directed segment is said to be from the initial point to the terminal point.*

Notation : The directed segment whose initial point is A and whose terminal point is B is denoted by the symbol $\overrightarrow{AB}$; the arrow on the top emphasizes that a direction is involved. Often, the arrow on the top is replaced by a ' bar ', as in $\overline{AB}$ (to be read as A-B-bar), it being understood that the first mentioned point, A, is the initial point and the second mentioned point B, is the terminal point.

The initial and terminal points of a directed segment are together called its **end-points.** Note that $\overline{AB}$ and $\overline{BA}$ are to be thought of as different directed segments. The direction of $\overline{AB}$ is from A to B; the direction of $\overline{BA}$ is from B to A. Diagrammatically, the initial and terminal points are distinguished by putting an ' arrow-head ' at the terminal point of the segment, or on the segment itself, as illustrated in Fig. 8·1.

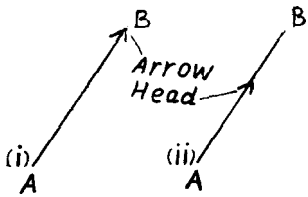


Fig 8.1

The directed segments that may be the object of our investigation at any

one time may be all coplanar, or they may not be coplanar but may lie in three-dimensional space.

(2) **Vector and Scalar Quantities** : In physics, we come across a number of physical quantities, such as force, velocity, momentum, electric field intensity, and magnetic field intensity, all of which have both a **magnitude** and **direction**, and so they cannot be represented or specified completely by giving their magnitudes only. They can, however, be represented by directed segments, and are called **vector quantities**, or simply, **vectors**.

In contrast with vectors, other physical quantities like mass, kinetic energy, potential energy, electric charge, temperature and distance, which have only magnitude, but no direction, can each be represented or specified completely by a single number, and are called **scalar quantities**, or simply, **scalars**.

Note that the specification of the magnitude of both vectors and scalars requires the selection of a unit of measurement such as kilogram (mass), meter (distance), erg (energy), or Newton (force).

Although a vector cannot be conveniently represented or specified by a single number, we can define certain **operations** with vectors which are similar to the usual operations with numbers; for example, we can define the ' sum ' of two vectors and a (scalar) ' multiple ' of a vector and then construct a new ' Algebra ', called ' **Vector Algebra** ', for their study and manipulations. This Algebra turns out to be similar in some respects to the ordinary algebra of numbers but quite different from it in some other respects. Since scalars are completely respresentable by numbers, there is no need to construct a separate Algebra of scalars. In fact, vectors could be defined to be those physical quantities whose mathematical represention obey the rules of Vector Algebra. Thus, not all physical quantities which have both magnitude and direction can be treated as vectors, even though they can be represented by directed segments; for example, the rotational motion of a three-dimensional body, fixed at one point, around an axis passing through the point.

This leads us to the following two definitions :

(i) **Vectors. Definition** : *Physical quantities having magnitudes (with a unit of measurment), directions and whose mathematical representations obey certain rules (known as rules of Vector Algebra) are called vector quantities.*

(ii) Scalar. Definition : *Quantities which have only magnitude but no direction are called scalar quantities or scalars.*

(iii) Notation for a vector : It is a common practice to denote a vector quantity say $\overrightarrow{AB}$ by a single letter printed in bold face type say, $\boldsymbol{u}$, or written with a bar on its top, for example, say, $\bar{u}$, or written with a wavy line under it, for example, say, $\underset{\sim}{u}$ (to be read as yootilde). Its magnitude is then denoted by $|\boldsymbol{u}|$, $|\bar{u}|$, or $|\underset{\sim}{u}|$ respectively. Thus if $\overrightarrow{AB} = \boldsymbol{u}$, then $|\overrightarrow{AB}| = AB = |\boldsymbol{u}|$.

(3) Equivalent directed segments and equal vectors: A directed segment such as $\overrightarrow{AB}$, has (i) **length** or **magnitude**, and (ii) **direction**. The **magnitude** of the directed segment $\overrightarrow{AB}$ is the distance between its two end-points A and B and is thus always non-negative. The magnitude of the directed segment $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$, or by just AB. The **direction** of the directed segment $\overrightarrow{AB}$ is the direction "from A to B", as indicated by the bar written on the top of AB or by an arrow-head on the directed segment.

Note that the concept of direction of a directed segment (and perhaps that of a directed segment itself) breaks down if the initial and terminal points of the segment coincide. Such a segment is called a **zero** or **null segment**; its magnitude is zero. Similarly, in physical applications, it is necessary to talk of a vector quantity whose magnitude could happen to be zero in a special case; a vector of zero magnitude is said to be a **zero vector** or a **null vector**. In this case only, one cannot talk of the direction of a vector. We now look at the concept of the direction of a directed segment and give the following definitions.

Definition 1. Two directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to have **the same direction** if, and only if, they lie either (i) along the same line or (ii) along parallel lines and have the same sense.

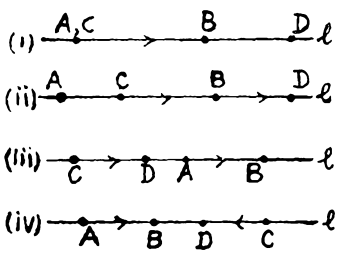


Fig. 8·2

Case (1) : If A, B, C, D are collinear, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to have the same direction if, and only if, B and D lie on the same side of A and C respectively. Thus in Fig. 8·2 (i), 8·2 (ii) and 8·2 (iii) the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are the same, whereas in 8·2 (iv), $\overrightarrow{AB}$ and $\overrightarrow{CD}$ do not have the same direction.

Case (2) : If A, B, C, D are not collinear and $\overline{AB}$, $\overline{CD}$ lie in distinct parallel lines, then $\overline{AB}$ and $\overline{CD}$ are said to be parallel to each other. The two initial points A, C of the two directed segments $\overline{AB}$, $\overline{CD}$ determine a unique straight line say l. The two parallel lines along which the segments lie determine a plane, say $\propto$. Clearly, the line l lies in the plane $\propto$ and divides the plane into two 'half-planes'. The directions of $\overline{AB}$ and $\overline{CD}$ are then defined to be the same if, and only if, the terminal points B and D lie in the same half-plane of $\propto$, that is, lie on the same side of l Fig. 8·3 (i) illustrates this case. The directions of $\overline{AB}$ and $\overline{CD}$ are not the same if B and D lie in different half-planes, i. e., lie on opposite sides of the line l. Fig. 8·3 (ii) illustrates this case.

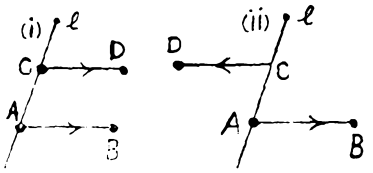


Fig. 8·3

Definition 2. *If two directed segments lie either along the same line or along parallel lines but do not have the same direction, they are said to have* ***opposite directions.***

Definition 3. Equivalent Directed Segments. *If two directed segments $\overline{AB}$ and $\overline{CD}$ have the same magnitude and direction, they are said to be* ***equivalent*** *or equal.*

We thus see that a directed segment can be specified by giving three of its characteristics : (a) magnitude, (b) direction, and (c) either the initial point or the terminal point. Equivalent directed segments are those having the same magnitude and direction, but not necessarily the same initial and terminal points.

Definition 4. Equal Vectors. *Two vectors are said to be* ***equal*** *if the directed segments representing them are equivalent.*

Two vectors are said to **have the same direction** if the directed segments representing them have the same direction. Clearly, two vectors are equal if, and only if, they have the same magnitude and direction. In the sense of this definition, we can talk of the equality of, say, a force vector and a velocity vector. If this seems undesirable, we could impose the additional condition that the two vectors must be of the same physical type or nature and must have the same unit of measurement.

(4) Free, localised, unit, zero vectors; negative of a vector; collinear and coplanar vectors.

(i) Free Vector : In many physical situations, a physical vector quantity can be represented by a directed segment in such a way that two directed segments which are equivalent can be taken or used to represent the same vector quantity. In other words, the magnitude and direction, but not the initial point, of the directed segment, representing the vector quantity are significant. Thus we have the following :

Definition of a Free Vector : *If we can choose the initial point A of a vector $\overline{AB}$ anywhere in space then the vector $\overline{AB}$ is called a free vector.*

(ii) Localized Vector : In contrast with free vectors, this arbitrary choosing of the initial point cannot be done for some other vector quantities such as a system of forces acting on a rigid body when the motion is not a pure translation. This leads us to the following.

Definition of a Localised Vector : *If the initial point A of a vector $\overline{AB}$ can be chosen anywhere in the line AB, then the vector $\overline{AB}$ is called a localised (or bound or line) vector.* In the case of a system of forces acting on a rigid body, however, two directed segments which are not only equivalent but also lie in the same line can be considered to represent the same forces; this is often expressed by saying that a force can be moved along its line of action without making any difference, or that a force is a **line vector.**

In what follows, we shall be dealing mainly with free vectors and so, henceforth the term ' vector ' shall mean a ' free vector ', unless otherwise stated. Note that even a free vector quantity can have a physical point of application; only, in its representation by a directed segment, the point of application does not play any role.

(iii) Unit Vector. Definition : *If the vector $\bar{u}$ is such that $|\bar{u}| = 1$, that is $\bar{u} = \overline{AB}$ and $|\overline{AB}| = 1$, then $\bar{u}$ is said to be a* ***unit vector.***

Thus, a unit vector has a unit magnitude.

(iv) Zero Vector. Definition : *If the vector $\bar{u}$ is such that $|\bar{u}| = 0$, then $\bar{u}$ is said to be a zero vector or a null vector.*

A zero vector thus has zero magnitude. It is often denoted by $\bar{0}$, or when no confusion can arise, by just 0.

(v) **Negative of a Vector. Definition** : *A vector whose magnitude is equal to the magnitude of the given vector* $\bar{u}$ *but is directed in a sense opposite to that of* $\bar{u}$ *is called the negative of* $\bar{u}$ *and is written as* $-\bar{u}$.

(vi) **Collinear Vectors. Definition** : *Vectors are said to be collinear if they lie either* (*i*) *along the same line or* (*ii*) *along parallel lines.*

(vii) **Coplanar Vectors. Definition.** *Vectors are said to be coplanar if they are all parallel to one and the same plane.*

Note that coplanar vectors need not lie in one and the same plane.

(5) **Addition of Vectors : Triangle, Parallelogram and Polygonal Laws.**

Consider now a translational motion of a body or of a system of rigidly connected particles. that is, motion in which the displacements of all the particles are the same in magnitude and direction. Clearly, such motion can be represented by a **single** directed segment, and further, any directed segment, equivalent to a given one, will do just as well to represent the motion. Thus, such a motion can be considered to be a **free vector**. Consider next some particular particle P of the body. Suppose that when the body first moves, the particle P moves from a position A to the position B. This motion or displacement of the body can. then, be represented by the directed segment $\overline{AB}$. If the body moves **further** in such a way that P moves from B to C, then this additional displacement can be represented by the directed segment $\overline{BC}$. The net or combined result of these two displacements is that P moves from A to C (see Fig. 8·4), and this net displacement, in turn, can be represented by the directed segment $\overline{AC}$. This particular example suggests a general definition of the **addition** of two free vectors.

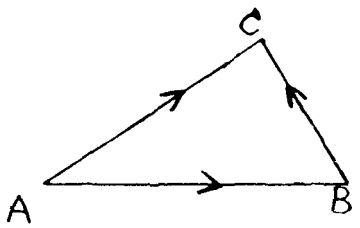


Fig 8·4

Definition : The sum of two free vectors $\bar{u}$ and $\bar{v}$ is denoted by $\bar{u} + \bar{v}$ and is defined as follows. Let a directed segment $\overline{AB}$ represent the first vector, namely, $\bar{u}$; let the **second** vector, $\bar{v}$, be represented by the directed segment $\overline{BC}$. Thus, the **initial** point of the segment representing the **second** vector is to coincide with the **terminal** point of the segment

representing the first vector. Then the free vector represented by the directed segment $\overline{AC}$ is defined to be the sum $\bar{u} + \bar{v}$ of the two free vectors $\bar{u}$ and $\bar{v}$ (see Fig. 8·5). In case C coincides with A [see Fig. 8·5 (d)], the sum of $\bar{u}$ and $\bar{v}$ will be the zero or the null vector $\bar{0}$.

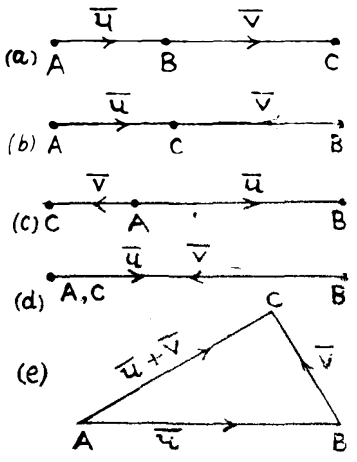


Fig. 8·5

We do not distinguish between a free vector and a directed segment representing it and we treat directed segments as if they are vectors. The sum of two vectors is given by the triangle Fig. 8·5 (*e*) and parallelogram rules (Fig. 8·6) given below.

(i) Triangle law of addition : *If we construct a* $\triangle ABC$ *such that* $\overline{AB} = \bar{u}$, $\overline{BC} = \bar{v}$ *then* $\overline{AC} = \bar{u} + \bar{v}$.

(ii) Parallelogram law for addition of two vectors : If we construct a parallelogram $ABCD$ such that $\overline{AB} = \bar{u}$ and $\overline{AD} = \bar{v}$, then we have $\overline{AC} = \bar{u} + \bar{v}$. [Fig. 8·6]. This law is obvious since $\overline{AD} = \overline{BC}$ and from the triangle law $\overline{AB} + \overline{BC} = \overline{AC} = \bar{u} + \bar{v}$. Note that in special cases Fig. 8.5 (a) to (d) both the triangle and parallelogram involved degenerate into a straight line.

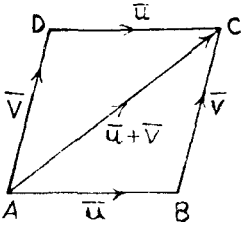


Fig. 8·6

Note : The sum $\bar{u} + \bar{v}$ is called the resultant of the two vectors $\bar{u}$ and $\bar{v}$. The method of obtaining the magnitude and direction of the resultant of two vectors is shown in the Illustrative example no. 1.

(iii) Polygonal Law, i. e., generalization of triangle law for addition. For finding the sum of two vectors, we constructed a triangle whose two sides, taken in order, represented the two vectors and their addition was represented by the third side in the proper sense. For finding the sum of three vectors $\bar{u}$, $\bar{v}$, $\bar{w}$ we choose three directed segments representing the three given vectors to form the three consecutive sides of a quadrilateral and then complete the quadrilateral by joining the initial point of the first segment or side to the terminal point of the last

segment or side ; this last drawn directed segment or 'closing' side represents the desired sum. Likewise the sum of any number, say, n of vectors can be obtained by constructing, in general, a polygon of $(n+1)$ sides. Hence this method is called the polygonal Law or Polygonal Rule of addition of vectors. **Note**, however, that the polygon may not be a convex one, its sides may cross each other even in the case of three vectors and it could be a degenerate one. See Fig. 8·7 (c) where the sum is marked with a double arrow.

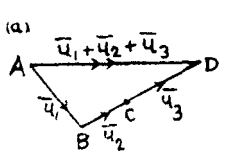


Fig. 8·7 (a)

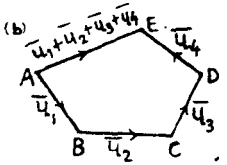


Fig. 8·7 (b)

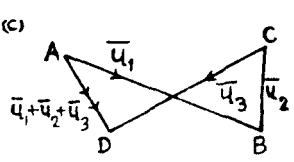


Fig. 8·7 (c)

Remark : It is obvious that the sum of a vector and its negative vector is a zero vector. Thus $\bar{u}+(-\bar{u})=\bar{0}$.

(6) **Properties of Addition of Vectors** : We now show that the operation of addition of vectors has some very simple but important properties, analogous to properties of the operation of addition of numbers.

Property I : Commutative Property : *Addition of vectors is commutative, that is, for any two vectors* $\bar{u}$, $\bar{v}$, $\bar{u}+\bar{v}=\bar{v}+\bar{u}$.

Proof : Let $\bar{u}=\overline{AB}$ and $\overline{AC}=\bar{v}$. Complete the parallelogram $ABDC$. (Fig. 8·8)

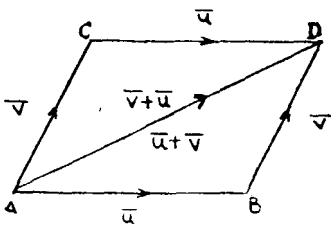


Fig. 8·8

$\because$ AB is equal and parallel to CD.

$\therefore$ $\overline{AB}=\overline{CD}$, $\therefore$ $\bar{u}=\overline{CD}$.

Similarly, since AC is equal and parallel to BD.

$\therefore$ $\overline{AC}=\overline{BD}$, $\therefore$ $\bar{v}=\overline{BD}$.

From $\triangle ABD$, $\overline{AB}+\overline{BD}=\overline{AD}$,

i. e., $\overline{AD}=\bar{u}+\bar{v}$. ... (i)

Also from $\triangle ACD$, $\overline{AC}+\overline{CD}=\overline{AD}$, i. e., $\overline{AD}=\bar{v}+\bar{u}$. ... (ii)

$\therefore$ From (i) and (ii), we have $\bar{u}+\bar{v}=\bar{v}+\bar{u}$.

The property is also immediately obvious from the parallelogram law as the parallelogram can be imagined to be constructed with AB and AC as adjacent sides having AD as the diagonal (Fig. 8·9) giving the result $\overline{AD} = \bar{u} + \bar{v}$.

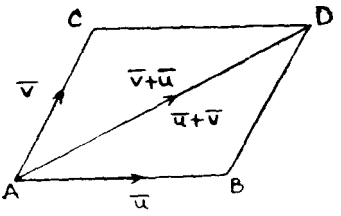


Fig. 8·9

Similarly the same parallelogram can be imagined to have been constructed with AC and AB as adjacent sides having the same diagonal AD giving the result $\overline{AD} = \bar{v} + \bar{u}$.

Hence it follows that $\bar{u} + \bar{v} = \bar{v} + \bar{u}$.

Property II : Associative Property : Addition of vectors is associative.

For any three vectors $\bar{u}$, $\bar{v}$, $\bar{w}$, $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$.

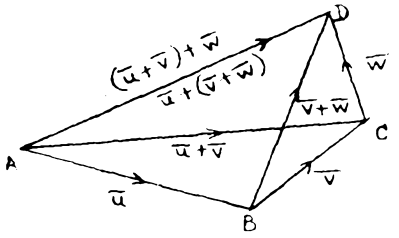


Fig. 8·10

Let $\bar{u} = \overline{AB}$, $\bar{v} = \overline{BC}$, $\bar{w} = \overline{CD}$. Join AC, AD and BD.

From $\triangle ABC$, we have $\overline{AB} + \overline{BC} = \overline{AC}$. i. e., $\bar{u} + \bar{v} = \overline{AC}$; and from $\triangle ACD$, we have $\overline{AC} + \overline{CD} = \overline{AD}$,

i. e., $(\bar{u} + \bar{v}) + \bar{w} = \overline{AD}$. ... (i)

Again from $\triangle BCD$, we have $\overline{BC} + \overline{CD} = \overline{BD}$, i. e., $\bar{v} + \bar{w} = \overline{BD}$; and from $\triangle ABD$, we have $\overline{AB} + \overline{BD} = \overline{AD}$, i. e., $\bar{u} + (\bar{v} + \bar{w}) = \overline{AD}$. ... (ii)

From (i) and (ii), we get

$$(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$$

and we write $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w}) = \bar{u} - \bar{v} + \bar{w}$.

(7) **Subtraction of vectors** : Once addition of vectors and the negative of a vector have been defined, it is a simple matter to define the subtraction of two vectors i. e., the difference between two vectors, in terms of addition and the negative.

Subtraction : Definition : *The difference of two vectors* $\bar{u}$, $\bar{v}$, *taken* ***in that order****, is defined to be the* ***sum*** $\bar{u} + (-\bar{v})$, *that is, the sum of the* ***first*** *vector and the* ***negative*** *of the* ***second****, and is denoted by* $\bar{u} - \bar{v}$.

Geometrical Interpretation of the subtraction of two vectors using the parallelogram Law.

Let in the parallelogram $ABCD$, $\overline{AB} = \bar{u}$ and $\overline{AD} = \bar{v}$ so that $\overline{AC} = \bar{u} + \bar{v}$.

Produce DA to D_1 so that $DA = AD_1$ and hence $\overline{AD_1} = -\bar{v}$. Complete the parallelogram AD_1C_1B so that $\overline{AC_1} = \bar{u} - \bar{v}$.

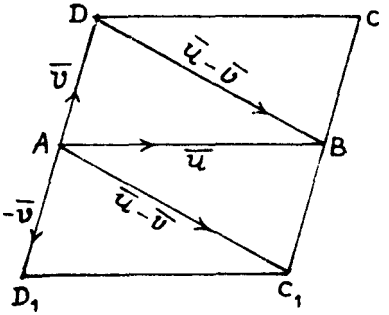


Fig. 8·11

Since $ADBC_1$ is evidently a parallelogram we have $\overline{AC_1} = \overline{DB}$.

$$\therefore \quad \overline{DB} = \bar{u} - \bar{v}.$$

Thus if two vectors are represented in magnitude and direction by the two adjacent sides of AB, AD of a parallelogram $ABCD$, then the diagonal AC drawn outward through their point of intersection represents the sum of the vectors whereas the other diagonal DB represents in magnitude and direction the difference of the two vectors.

Remark : Since in a parallelogram, the sum of the squares on its diagonals is equal to the sum of the squares of its four sides, we have

$$AC^2 + DB^2 = AB^2 + BC^2 + CD^2 + DA^2 = 2AB^2 + 2AD^2$$

$$\therefore \quad |\bar{u}+\bar{v}|^2 + |\bar{u}-\bar{v}|^2 = 2|u|^2 + 2|\bar{v}|^2.$$

(8) **Scalar Multiplication of Vectors** : Suppose $\bar{u}$ is a vector; consider the sum $\bar{u} + \bar{u}$; clearly, the magnitude of the vector $\bar{u} + \bar{u}$ is **twice** the magnitude of $\bar{u}$ and its direction is the same as that of $\bar{u}$. Because of these facts and because of the close similarity in the properties of addition of numbers and addition of vectors, it is natural to denote the sum $\bar{u} + \bar{u}$ by $2\bar{u}$. Similarly, $\bar{u} + \bar{u} + \bar{u}$ is denoted by $3\bar{u}$; the magnitude of $3\bar{u}$ will be **thrice** that of $\bar{u}$. More generally, if n is a positive integer, $n\bar{u}$ can be defined as the sum $\bar{u} + \bar{u} + \ldots\ldots n$ times; we have $|n\bar{u}| = n|\bar{u}|$ and $n\bar{u}$ has the same direction as that of $\bar{u}$; we can define $1\bar{u}$ to be $\bar{u}$ itself and $0\bar{u}$ to be the zero vector $\bar{0}$.

These considerations lead us to the following definition of the multiplication of a vector by a real number or a scalar.

Definition : Let $\bar{u}$ be a vector and $\propto$ a real number. Then the **product of $\propto$ and $\bar{u}$, or the multiple of the vector $\bar{u}$ by the real number $\propto$,** denoted by $(\propto)\,\bar{u}$, or by just $\propto \bar{u}$, is the vector whose magnitude is $|\propto|$ times the magnitude of $\bar{u}$, and whose direction is the same as that of $\bar{u}$ if $\propto > 0$, and opposite to that of $\bar{u}$ if $\propto < 0$. If $\propto = 0$, $\propto . \bar{u}$ is defined to be the zero vector $\bar{0}$.

The term ' scalar ' is sometimes used for a real number when the latter multiplies a vector, and the operation of multiplying a vector by a real number is called ' **scalar multiplication** '. In some physical situations, a significance can be attached to the product of a scalar physical quantity by a vector physical quantity; for example, force is the product of mass and acceleration.

To illustrate scalar multiplication further, suppose ABC is a triangle, D is the mid-point of seg BC and G is the centroid of the triangle (see Fig. 8·12). Suppose $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{x}$, $\bar{y}$, $\bar{z}$ are vectors represented by $\overrightarrow{BC}$, $\overrightarrow{BD}$, $\overrightarrow{DB}$, $\overrightarrow{AG}$, $\overrightarrow{AD}$ and $\overrightarrow{DG}$ respectively. Then we have $\bar{u} = 2\bar{v}$; $\bar{v} = \frac{1}{2}\bar{u}$, $\bar{w} = -\frac{1}{2}\bar{u}$, $\bar{x} = \frac{2}{3}\bar{y}$; $\bar{z} = -\frac{1}{2}\bar{x}$.

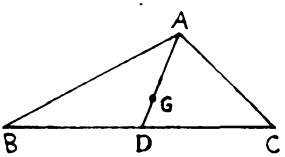


Fig. 3·12

Remarks:

(i) $\bar{u} = |\bar{u}|\,\hat{u}$, where $\hat{u}$ denotes a unit vector in the same direction as that of $\bar{u}$; thus any vector in a given direction may be expressed as a scalar multiple of a unit vector in that direction.

(ii) If a fixed non-zero vector $\bar{v}$ is selected, all vectors, either in the same direction as $\bar{v}$ or in a direction opposite to that of $\bar{v}$, can be expressed as a scalar multiple of $\bar{v}$.

EXERCISE 8 (*a*)

(1) State whether the following statements are true or false :

(i) Only those vectors which lie along the same line are collinear vectors.

(ii) Only those vectors which lie in one and the same plane are coplanar vectors.

(iii) All physical quantities which have both magnitude and direction can be treated as vectors.

(iv) $m\bar{u}$ has the same direction as that of $\bar{u}$ for any scalar m.

(v) All collinear vectors can be expressed as scalar multiples of a single suitable vector.

(2) Given the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ show how to construct the vectors

(i) $\bar{a} + 3\bar{b}$, (ii) $\bar{a} + 3\bar{b} + 2\bar{c}$ and (iii) $\bar{a} + 3\bar{b} - 2\bar{c}$.

(3) State any three physical entities which (i) can be, (ii) cannot be considered as vectors.

(4) Fill in the gaps in the following :

(i) If PQR is a triangle, then $\overline{PQ} + \overline{QR} + \overline{RP} =$ .....

(ii) If $PQRS$ is a parallelogram, then $\overline{PQ} + \overline{PR} + \overline{PS} =$ .....

(iii) If M is the mid-point of seg AB, then

(a) $\overline{MA} + \overline{MB} =$ ....., (b) $\overline{AM} + \overline{BM} =$ .....

(iv) If $m < 0$, then $m\bar{u}$ is a vector of magnitude ..... and has direction of .....

(v) If O is the origin and $A = (0, 1)$, then $\overline{OA}$ is a ..... vector.

(vi) If $P = (4, 5)$ and $Q(5, 5)$, then $\overline{PQ}$ is a ..... vector.

(vii) If $\bar{u}$ is a non-zero vector, then the unit vector in the direction of $\bar{u}$ is .....

(viii) If $m(\angle AOB) = \pi/2$ in $\triangle AOB$ and D is the mid-point of AB, find k if

(i) $\overline{AD} = k\,\overline{AB}$, (ii) $\overline{AD} = k\,\overline{DB}$, (iii) $\overline{AD} = k\,\overline{BD}$.

(5) Complete the following statements by choosing the correct word from the given alternatives.

" Zero, unit, equal, coplanar, scalar multiple, negative ".

If $ABCDEF$ is a regular hexagon, then

(i) the three pairs of parallel vectors : $\overline{AB}$, $\overline{ED}$; $\overline{AF}$, $\overline{CD}$; $\overline{CB}$ and $\overline{EF}$ are pairs of ..... vectors;

(ii) $\overline{AB}$, $\overline{BC}$, $\overline{BE}$ and $\overline{CF}$ are ... .. vectors;

(iii) $\overline{AB} - \overline{ED}$ is a .....vector;

(iv) $\overline{AD}$ is a ..... of $\overline{BC}$;

(v) $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF} + \overline{FA}$ is a ..... vector;

(vi) magnitudes of the vectors $\overline{BD}$ and $\overline{FA}$ are ... ..;

(vii) $\overline{AB}$ is the ... .. of $\overline{DE}$.

ANSWERS

(1) All the statements except (v) are false.

(3) Vector quantities : Velocity of a particle, force acting on a particle, weight of a body etc.

Quantities which are not vectors : mass, kinetic energy of a body in motion, temperature etc.

(4) (i) $\bar{0}$; (ii) $2\overline{PR}$; (iii) (a) $\bar{0}$, (b) $\bar{0}$; (iv) $-m|\bar{u}|, -\bar{u}$;
(v) Unit, (vi) Unit, (vii) $\bar{u}/|\bar{u}|$, (viii) (i) $k = 1/2$,
(ii) $k = 1$, (iii) $k = -1$.

(5) (i) equal, (ii) coplanar, (iii) zero, (iv) scalar multiple, (v) zero, (vi) equal, (vii) negative.

(9) Position vectors : Suppose we consider a plane or three-dimensional space. Suppose further, that a particular point O of the plane or space is chosen as the **reference point** or the **origin.** For any point, say P, of the plane or space, if we choose O as the initial point and P as the terminal point, we get a directed segment $\overrightarrow{OP}$ or $\overline{OP}$.

Thus we have the following :

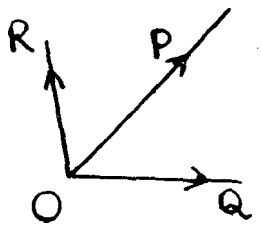


Fig. 8·13

Definition : *The position vector of a point P with respect to an origin O is the directed line segment* $\overline{OP}$ *and may be denoted by* $\bar{p}$. Clearly, the position vector of the point O (the origin) with respect to itself is a zero vector. Note that with a given point P, we can associate a unique position vector only when the origin has been chosen; by choosing different points as the origins, we can associate different position vectors with the same point P. However, usually, we assume that some one point has been chosen as the origin once and for all, and we talk about the position vectors of points, dropping the phrase ' with respect to the origin O'. A ' position vector ', strictly speaking, is not a vector quantity at all; it is not something ' physical ' like force, velocity, or displacement, but rather something purely geometrical or mathematical. It is, in fact, a special case of a directed segment, namely, when the initial point of the directed segment is a stipulated, fixed or given point, called the origin. In view of this, the question as to whether position vectors are free vectors or bound vectors does not arise. However, position vectors can be associated with free vectors as follows. As we have seen, a free vector is represented by any one from a set of directed

segments, all of which are equivalent to one another, and among these will be a **unique** one with the given origin O as its initial point. Conversely, a position vector, being a directed segment, represents a **unique** free vector. Thus, there is a one-to-one correspondence between free vectors and position vectors with respect to a given origin. Unlike this, there is no such one-to-one correspondence between directed segments and free vectors. This makes it very convenient to use position vectors rather than directed segments when dealing with free vectors, and often, one does not even distinguish between a position vector and the unique free vector represented by it and applies the operations of addition and scalar multiplication, defined for free vectors, to position vectors also. Because of the special nature of position vectors, we consider in detail some operations involving the Algebra of position vectors and the special case of the addition and scalar multiplication of position vectors.

(10) Algebra of Position Vectors.

(i) Addition of Position Vectors.

Definition : Let $\overrightarrow{OP}$, $\overrightarrow{OQ}$ be the position vectors of two points P, Q with respect to an origin O. Then their sum, denoted by $\overrightarrow{OP} + \overrightarrow{OQ}$, is the position vector $\overrightarrow{OR}$ of the point R, where $OPRQ$ is a parallelogram with seg OP and seg OQ as two of its sides (Fig. 8·14).

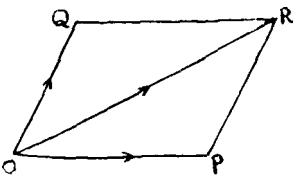


Fig. 8·14

(ii) Multiplication of a Position Vector by a Real Number.

Definition : Let $\overrightarrow{OP}$ be the position vector of a point P with respect to the origin O and let α be a real number. Then, the multiple of the position vector $\overrightarrow{OP}$ by the real number α is denoted by $\alpha\,\overrightarrow{OP}$ and is defined to be the position vector $\overrightarrow{OR}$ of the point R such that (i) O, P, R are collinear; (ii) $|\overrightarrow{OR}| = |\alpha|\,|\overrightarrow{OP}|$; and (iii) if α is positive, P and R are on the same side of O and if α is negative, P and R are on opposite sides of O. If $\alpha = 0$, then $\alpha\,\overrightarrow{OP}$ is the zero position vector, that is, the position vector $\overrightarrow{OO}$ of the point O with respect to O itself. (See Fig. 8·15).

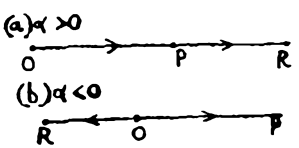


Fig. 8·15

(iii) Expression of a given vector say, $\bar{u} = \overline{PQ}$, in terms of the position vectors of its end points P and Q.

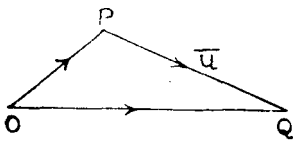


Fig. 8.16

Let the position vectors of P and Q referred to the origin O be $\bar{p}$ and $\bar{q}$.

From the triangle OPQ we have

$\overline{OP} + \overline{PQ} = \overline{OQ}$, i. e., $\bar{p} + \overline{PQ} = \bar{q}$.

$\therefore \overline{PQ} = \bar{q} - \bar{p}$.

In other words $\overline{PQ}$ = (the position vector of Q) − (the position vector of P).

(11) Section Formula : The following result known as the **Section Formula** is very useful in solving some problems, especially on collinearity of points and concurrency of lines in geometry, by vector methods.

Theorem : If A, B be two points with position vectors $\bar{a}, \bar{b}$ respectively, with reference to some point O as origin, and if P divides seg AB in the ratio $m : n$ (internally or externally) and if $\bar{p}$ be its position vector, then

$$\bar{p} = \frac{m\bar{b} + n\bar{a}}{m + n}. \qquad \ldots (1)$$

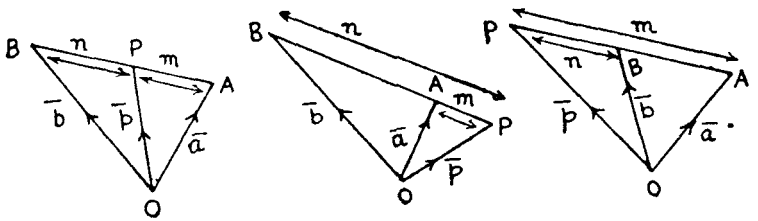


Fig. 8·17

We observe that the ratio $m : n$ is positive if P divides seg AB internally and negative if it divides seg AB externally. Further, when $m : n$ is negative, $|m/n| < 1$ if A is between P and B and $|m/n| > 1$ if B is between A and P. Further m/n can never be -1, i. e., $m + n \neq 0$, since a segment AB cannot be divided externally in the numerical ratio $1 : 1$ at a finite point. (See Fig. 8·17).

To prove the result, we observe that

$$\frac{|\overline{AP}|}{|\overline{PB}|} = \left|\frac{m}{n}\right|$$

and that $\overline{AP}$, $\overline{PB}$ have the same direction if $m/n > 0$, and they have opposite directions if $m/n < 0$. Hence, in all cases. $\overline{AP} = (m/n)\,\overline{PB}$.

$$\therefore \quad \bar{p} - \bar{a} = \frac{m}{n}(\bar{b} - \bar{p}), \text{ i. e., } n(\bar{p} - \bar{a}) = m(\bar{b} - \bar{p}).$$

$$\therefore \quad \bar{p} = \frac{m\bar{b} + n\bar{a}}{m+n}, \text{ provided } m + n \neq 0, \text{ i. e., } m : n \neq -1. \quad \ldots (1)$$

Cor : If P be the mid-point of $\overline{AB}$ we can take $m = n = 1$,

and $\quad \therefore \bar{p} = \frac{\bar{a} + \bar{b}}{2}$.

Sometimes the ratio $m : n$ is denoted by a single symbol λ, say.

We then have $$\bar{p} = \frac{\lambda\bar{b} + \bar{a}}{\lambda + 1} \quad \ldots (2)$$

as the position vector of the point which divides the segment AB in the ratio λ.

Note : It may be noted that in the numerator of (1), m, which corresponds to $\overline{AP}$, multiplies $\bar{b}$ and n, which corresponds to $\overline{PB}$, multiplies $\bar{a}$. The figure 8·18 will be of help in remembering the result.

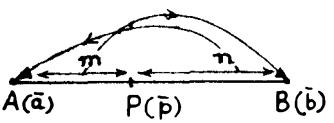


Fig. 8·18

(12) Collinearity of points. Theorem : The necessary and sufficient condition for the three distinct points A, B, C, with position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ respectively, to be collinear is that there should exist three non-zero scalars l, m, n such that $l\bar{a} + m\bar{b} + n\bar{c} = \bar{0}$, where $l + m + n = 0$.

Part 1. To prove that the condition is necessary, let us assume that $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ are collinear. Let $C(\bar{c})$ divide seg AB in the ratio $m : l$. Then by the section formula, we have

$$\bar{c} = \frac{l\bar{a} + m\bar{b}}{l + m}, \text{ i. e., } l\bar{a} + m\bar{b} = (l + m)\,\bar{c}.$$

Writing $l + m = -n$, we get $l\bar{a} + m\bar{b} + n\bar{c} = \bar{0}$, where $l + m + n = 0$.

Part 2. To prove that the condition is sufficient, we assume that $l\bar{a} + m\bar{b} + n\bar{c} = \bar{0}$ where $l + m + n = 0$.

We, therefore, get $l\bar{a} + m\bar{b} = -n\bar{c} = (l+m)\bar{c}$, since $-n = (l+m)$.

$$\therefore \quad \bar{c} = \frac{l\bar{a} + m\bar{b}}{l+m}, \text{ i. e., } C \text{ divides seg } AB \text{ in the ratio } m : l.$$

$\therefore$ A, B, C are collinear.

Note. The result proved in part (2) embodies the converse of the Section Formula.

(13) Applications of the Section Formula and its Converse :

(i) Concurrency and Collinearity : The section formula and its converse are very useful in applying vector methods to prove geometrical problems on concurrency of lines and collinearity of points. Thus to show that a certain set of lines is concurrent, we have essentially to show that all of them pass through a common point; this can be done by actually computing the position vector of this point. Similarly to show that three points A, B, P are collinear, it is enough to show that one of them, say P, is such that its position vector is the same as that of a point which divides the join of the other two in some ratio.

(ii) Vector Equation of a Line. If $\bar{a}$, $\bar{b}$ are the position vectors of the points A, B and $\bar{r}$ be the position vector of any point R on the line AB dividing seg AB in the ratio $m : n$ then we have by the section formula

$$\bar{r} = \frac{n\bar{a} + m\bar{b}}{n+m}, \; m + n \neq 0.$$

This, in fact, gives the vector equation of the line AB. For convenience, if we put $\frac{n}{n+m} = t$ and hence $\frac{m}{n+m} = 1 - t$, then the vector equation of the line takes the form $\bar{r} = t\bar{a} + (1-t)\bar{b}$, or $\bar{r} = \bar{b} + t(\bar{a} - \bar{b})$ where t is a parameter. We do not propose to discuss the vector form of the equation of a line and use it for any discussion in this book.

(14) Resolution of Vectors, Components and Ordered Pairs.

Theorem : If $\bar{a}$ and $\bar{b}$ are non-zero and non-parallel vectors and $\bar{r}$ is a vector which lies in a plane parallel to $\bar{a}$ and $\bar{b}$ then $\bar{r}$ can be uniquely expressed as a linear combination of $\bar{a}$ and $\bar{b}$ in the form

$\bar{r} = x\bar{a} + y\bar{b}$ where x and y are scalars.

Proof : $\because$ $\bar{r}$ lies in a plane parallel to $\bar{a}$ and $\bar{b}$,

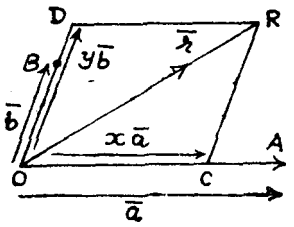


Fig 8·19

$\therefore$ we can find co-planar directed segments $\overline{OR}$, $\overline{OA}$ and $\overline{OB}$ representing the vectors $\bar{r}$, $\bar{a}$ and $\bar{b}$ respectively.

Through R draw $RC \parallel OB$ intersecting the line OA in C.

Similarly draw $RD \parallel OA$ intersecting the line OB in D.

$\because$ $\overline{OC}$ and $\overline{OA}$ are collinear,

$\therefore$ there exists a scalar x such that $\overline{OC} = x\,\overline{OA} = x\,\bar{a}$.

Similarly, since $\overline{OD}$ and $\overline{OB}$ are collinear, there exists a scalar y such that $\overline{OD} = y\,\overline{OB} = y\,\bar{b}$.

By the parallelogram, — law we have

$\overline{OR} = \overline{OC} + \overline{OD}$, i. e. $\bar{r} = x\,\bar{a} + y\,\bar{b}$.

The uniqueness of this expression follows from the fact that the parallelogram having OR as a diagonal and the sides through O parallel to $\bar{a}$ and $\bar{b}$ is unique.

Thus, if we have $\bar{r} = x\,\bar{a} + y\,\bar{b}$ and $\bar{r} = x'\,\bar{a} + y'\,\bar{b}$ then $x = x'$ and $y = y'$.

Definition: *If $\bar{r} = x\,\bar{a} + y\,\bar{b}$ then $x\,\bar{a}$ and $y\,\bar{b}$ are called the components of $\bar{r}$ in the directions of $\bar{a}$ and $\bar{b}$ respectively and the vector $\bar{r}$ is said to be resolved or decomposed in the directions of $\bar{a}$ and $\bar{b}$.*

Vectors and Ordered Pairs : We have proved in the above Theorem that if $\bar{a}$ and $\bar{b}$ are two given non-zero, non-parallel vectors then a vector $\bar{r}$ coplanar with $\bar{a}$ and $\bar{b}$ can be expressed uniquely in the form

$$\bar{r} = x\,\bar{a} + y\,\bar{b}.$$

Thus for given vectors $\bar{a}$, $\bar{b}$ there exists one-to-one correspondence between the coplanar vectors $\bar{r}$ and the ordered pairs (x, y) of real numbers. This ordered pair may be written horizontally as $[x, y]$ and it may then be called a row vector, or it may be written vertically as $\begin{bmatrix} x \\ y \end{bmatrix}$ and it may then be called a column vector. In fact, (x, y) are the co-ordinates of the point R with OA and OB as co-ordinate axes.

(15) **Rectangular Resolution of a Vector** : If $\bar{a}$ and $\bar{b}$ are two mutually perpendicular vectors then we can choose a pair of mutually perpendicular lines $X'OX$ and $Y'OY$ parallel to $\bar{a}$ and $\bar{b}$ respectively forming a frame of rectangular co-ordinate axes. Let $\bar{i}$ and $\bar{j}$ be the unit vectors along the x and y axes respectively. If $\bar{r} = \overline{OR}$ is any vector coplanar with $\bar{a}$ and $\bar{b}$ and hence with $\bar{i}$ and $\bar{j}$ then $\overline{OR} = x\bar{i} + y\bar{j}$ and the ordered pair associated with $\overline{OR}$ will be (x, y), i. e., identical with the rectangular co-ordinates (x, y) of the point R.

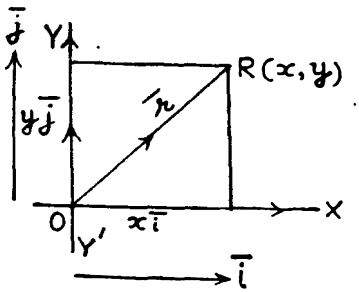


Fig. 8.20

This fact provides an important connection between vector algebra and co-ordinate geometry.

We have $|\bar{r}| = |\overline{OR}| = |x\bar{i} + y\bar{j}| = \sqrt{x^2+y^2}$

and hence $OR = \sqrt{x^2+y^2}$.

If θ is the angle between the x-axis and $\overline{OR}$, then θ is given by

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$

(16) **Sum, difference, and scalar multiplication of vectors when expressed as ordered pairs.**

If $\bar{a} = (x_1, y_1)$ and $\bar{b} = (x_2, y_2)$ then by taking $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ and using the definition of vector addition we have

(i) $\bar{a} + \bar{b} = (x_1 + x_2, y_1 + y_2)$,

(ii) $\bar{a} - \bar{b} = (x_1 - x_2, y_1 - y_2)$.

(iii) $m\bar{a} = (mx_1, my_1)$,

(iv) $-\bar{a} = (-x_1, -y_1)$

(v) $m\bar{a} \pm n\bar{b} = (mx_1 \pm nx_2, my_1 \pm ny_2)$.

Thus, in particular, if $\bar{a} = (2, 3)$ and $\bar{b} = (6, -2)$,

then (i) $\bar{a} + \bar{b} = (2 + 6, 3 - 2)$, i. e., $(8, 1)$.

(ii) $\bar{a} - \bar{b} = (2 - 6, 3 + 2,)$, i. e., $(-4, 5)$.

(iii) $3\bar{a} + 4\bar{b} = [3\cdot2 + 4\cdot6, \quad 3\cdot3 + 4(-2)] = (30, 1)$.

We also note that

$\overline{AB} = \bar{b} - \bar{a} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$.

If the components of the ordered pairs are the rectangular co-ordinates of the corresponding points, we have

$AB = |\overline{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which is the usual distance formula in Co-ordinate Geometry.

Illustrative Examples.

Ex. 1. Find the magnitude and direction of two vectors $\bar{a}$ and $\bar{b}$ which include an angle $\propto$.

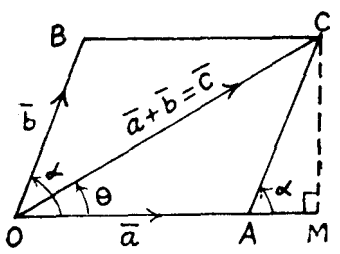


Fig. 8.21

Let $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$.
Complete the parallelogram $OACB$.
By the parallelogram law, we have
$\bar{a} + \bar{b} = \overline{OC} =$, say $\bar{c}$.
Let $m(\angle AOB) = \propto$
and let $m(\angle AOC) = \theta$.
Let $|\bar{a}| = a$, $|\bar{b}| = b$ and $|\bar{c}| = c$.
Let seg $CM \perp OA$ so that
$m(\angle MAC) = \propto$.
From $\triangle OAC$, we have by the cosine rule

$c^2 = a^2 + b^2 - 2ab \cos(\pi - \propto)$, i. e., $c^2 = a^2 + b^2 + 2ab \cos \propto$.

$\therefore$ the magnitude of the resultant of $\bar{a}$ and $\bar{b}$ is given by

$\sqrt{a^2 + b^2 + 2ab \cos \propto}$.

From $\triangle OCM$, we have $\tan \theta = \dfrac{MC}{OM} = \dfrac{b \sin \propto}{a + b \cos \propto}$.

Thus the angle made by the resultant with the direction of $\bar{a}$ is

$$\tan^{-1}\left(\frac{b \sin \propto}{a + b \cos \propto}\right).$$

Remarks.

(i) If $\bar{a} \parallel \bar{b}$ and both have the same sense then $\bar{a}$, $\bar{b}$ are called like parallel vectors including the angle zero.

In this case the magnitude of their resultant is given by

$c^2 = a^2 + b^2 + 2ab \cos 0 = a^2 + b^2 + 2ab = (a + b)^2$.

$\therefore \quad c = |a + b|$.

The direction of the resultant is the same as that of $\bar{a}$ and $\bar{b}$.

(ii) If $\bar{a} \parallel \bar{b}$ but they have opposite directions, then $\bar{a}$, $\bar{b}$ are called unlike parallel vectors including the angle π.

In this case the magnitude of their resultant is given by

$$c^2 = a^2 + b^2 + 2ab\cos\pi = a^2 + b^2 - 2ab = (a - b)^2.$$

$$\therefore \quad c = |a - b|.$$

The direction of the resultant is that of $\bar{a}$ if $a > b$ and is that of $\bar{b}$ if $a < b$. The resultant is obviously a zero vector if $a = b$.

(iii) If $\bar{a} \perp \bar{b}$, then $\alpha = \pi/2$ so that $\cos\alpha = 0$.

In this case, $c^2 = a^2 + b^2$.

$$\therefore \quad c = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}(b/a).$$

Ex. 2. If O, A, B are any three points, show that $\overline{OA} + \overline{OB} = 2\overline{OM}$, where M is the mid-point of seg AB.

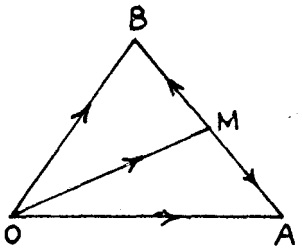


Fig. 8·22

We have $\overline{OA} + \overline{OB}$

$= (\overline{OM} + \overline{MA}) + (\overline{OM} + \overline{MB})$

$= (\overline{OM} + \overline{OM}) + (\overline{MA} + \overline{MB})$

$= 2\overline{OM} + \bar{0} = 2\overline{OM}$.

Alternative Method. Let $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$ and $\overline{OM} = \bar{m}$ so that $\bar{a}$, $\bar{b}$, $\bar{m}$ are the position vectors of A, B, M respectively, with respect to O as the origin.

By the section formula, we have

$$\bar{m} = \frac{1\cdot\bar{a} + 1\cdot\bar{b}}{1 + 1}, \text{ i. e, } \bar{a} + \bar{b} = 2\bar{m}.$$

$$\therefore \quad \overline{OA} + \overline{OB} = 2\overline{OM}.$$

Ex. 3. $ABCD$ is a quadrilateral and the lines joining the mid-points of its opposite sides intersect in P. Prove that

(i) P bisects each one of them

and (ii) $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 4\overline{OP}$, where O is any point.

Let L, M, N, Q be the mid-points of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Then

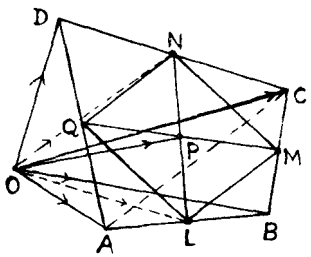


Fig. 8.23

$LM \parallel AC$, $QN \parallel AC$, $\overline{LM} = (1/2)\overline{AC}$, $\overline{QN} = (1/2)\overline{AC}$ and hence $LM \parallel QN$ and $\overline{LM} = \overline{QN}$. Therefore, $LMNQ$ is a parallelogram and hence $\overline{LN}$ and $\overline{MQ}$ bisect each other at P. Then, using the result in example 1,

$$\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD}$$

$$= (\overline{OA} + \overline{OB}) + (\overline{OC} + \overline{OD})$$

$$= 2\overline{OL} + 2\overline{ON}$$

$$= 2(\overline{OL} + \overline{ON}) = 2(2\overline{OP}) = 4\overline{OP}.$$

Alternative Method.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{l}, \bar{m}, \bar{n}, \bar{q}$ be the position vectors of A, B, C, D, L, M, N, Q respectively. Then by data, we have

$$\bar{l} = \frac{\bar{a} + \bar{b}}{2}, \bar{m} = \frac{\bar{b} + \bar{c}}{2}, \bar{n} = \frac{\bar{c} + \bar{d}}{2}, \bar{q} = \frac{\bar{d} + \bar{a}}{2}.$$

By the section formula, we have the position vector of the mid-point of seg LN as

$$\frac{1 \cdot \left(\frac{\bar{a} + \bar{b}}{2}\right) + 1 \cdot \left(\frac{\bar{c} + \bar{d}}{2}\right)}{1 + 1} = \frac{\bar{a} + \bar{b} + \bar{c} + \bar{d}}{4}.$$

Similarly, the position vector of the mid-point of seg MQ is

$$\frac{\bar{a} + \bar{b} + \bar{c} + \bar{d}}{4}.$$

Hence the segments LN and MQ bisect each other.

If P is their common point of bisection, with position vector $\bar{p}$, we have

$$\bar{p} = \frac{\bar{a} + \bar{b} + \bar{c} + \bar{d}}{4}.$$

$$\therefore \quad \bar{a} + \bar{b} + \bar{c} + \bar{d} = 4\,\bar{p}.$$

$$\therefore \quad \overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 4\,\overline{OP}.$$

Ex. 4. $ABCDEF$ is a regular hexagon. If $\overline{AB}$, $\overline{BC}$ represent vectors $\bar{a}, \bar{b}$, respectively, then show that

$\overline{CD} = \bar{b} - \bar{a}$, $\overline{DE} = -\bar{a}$, $\overline{EF} = -\bar{b}$, and $\overline{FA} = \bar{a} - \bar{b}$.

We have $AD = 2BC$ and $\overline{AD}$, $\overline{BC}$ have the same direction. Therefore, $\overline{AD} = 2\overline{BC} = 2\bar{b}$. Now $\overline{CD} = \overline{CA} + \overline{AD}$

$= (\overline{CB} + \overline{BA}) + \overline{AD}$

$= (-\bar{b} - \bar{a}) + 2\bar{b} = \bar{b} - \bar{a}$.

Also, $\overline{FA} = -\overline{CD} = -(\bar{b} - \bar{a}) = \bar{a} - \bar{b}$.

From the properties of a regular hexagon, we also have

$\overline{DE} = -\overline{AB} = -\bar{a}$, $\overline{EF} = -\overline{BC} = -\bar{b}$.

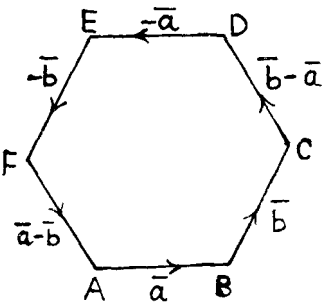


Fig. 8·24

Ex. 5. $\bar{a}, \bar{b}, \bar{c}$ are the position vectors of the three points A, B, C, respectively. The point P divides seg AB internally in the ratio $2 : 1$ and the point Q divides seg BC externally in the ratio $3 : 2$. Express $\overline{PQ}$ in terms of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

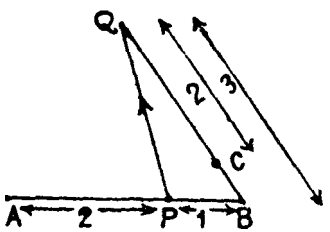


Fig. 8·25

Let $\bar{p}$ and $\bar{q}$ be the position vectors of P and Q respectively. Then by the section formula, we have

$$\bar{p} = \frac{\bar{a} + 2\bar{b}}{3} \text{ and } \bar{q} = \frac{2\bar{b} - 3\bar{c}}{2 - 3} = \frac{3\bar{c} - 2\bar{b}}{1}.$$

Now $\overline{PQ} = \bar{q} - \bar{p} = (3\bar{c} - 2\bar{b}) - \dfrac{(\bar{a} + 2\bar{b})}{3}$

$$= \frac{9\bar{c} - 8\bar{b} - \bar{a}}{3}.$$

Ex. 6. The position vectors $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{s}$ of the points P, Q, R, S are

$\bar{p} = 3\bar{a} + \bar{b} - 3\bar{c}$, $\bar{q} = 4\bar{a} + 3\bar{b} - 3\bar{c}$, $\bar{r} = 2\bar{a} - 7\bar{b} + 5\bar{c}$,

$\bar{s} = 4\bar{a} - 3\bar{b} + 5\bar{c}$ respectively.

Prove that $PQ \parallel RS$.

From the data, we have $\overline{PQ} = (4\bar{a} + 3\bar{b} - 3\bar{c}) - (3\bar{a} + \bar{b} - 3\bar{c}) = \bar{a} + 2\bar{b}$,

and $\overline{RS} = (4\bar{a} - 3\bar{b} + 5\bar{c}) - (2\bar{a} - 7\bar{b} + 5\bar{c}) = 2\bar{a} + 4\bar{b}$.

Thus $\overline{RS} = 2\bar{a} + 4\bar{b} = 2(\bar{a} + 2\bar{b}) = 2\overline{PQ}$.

$\therefore$ $PQ \parallel RS$.

Ex. 7. Show that the points P, Q, R with position vectors $\bar{p} = 2\bar{a} - \bar{b} + 3\bar{c}$, $\bar{q} = 3\bar{a} + 4\bar{b} - 5\bar{c}$ and $\bar{r} = \bar{a} - 6\bar{b} + 11\bar{c}$ respectively, are collinear and that P bisects seg RQ.

From the data, we have

$$\overline{PQ} = (3\bar{a} + 4\bar{b} - 5\bar{c}) - (2\bar{a} - \bar{b} + 3\bar{c}) = \bar{a} + 5\bar{b} - 8\bar{c}$$

and $\overline{QR} = (\bar{a} - 6\bar{b} + 11\bar{c}) - (3\bar{a} + 4\bar{b} - 5\bar{c}) = -2\bar{a} - 10\bar{b} + 16\bar{c}$.

Thus $\overline{QR} = -2(\bar{a} + 5\bar{b} - 8\bar{c}) = -2\overline{PQ}$;

i. e. $2\overline{PQ} = \overline{RQ}$.

$\therefore$ P, Q, R are collinear and P is the mid-point of RQ.

Ex. 8. If A, B, C, D, E are points such that $\overline{AB} + \overline{CD} = \overline{CB} + \overline{ED}$, then show that A and E coincide.

Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ be the position vectors of A, B, C, D, E respectively so that we have $\overline{AB} = \bar{b} - \bar{a}$, $\overline{CD} = \bar{d} - \bar{c}$, $\overline{CB} = \bar{b} - \bar{c}$ and $\overline{ED} = \bar{d} - \bar{e}$.

Then from the data, we have

$$\bar{b} - \bar{a} + \bar{d} - \bar{c} = \bar{b} - \bar{c} + \bar{d} - \bar{e}.$$

$\therefore \quad -\bar{a} = -\bar{e}$; i. e., $\bar{a} = \bar{e}$ and, therefore, A and E coincide.

Ex. 9. Show that the medians of a triangle are concurrent.

Let ABC be a triangle and let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of A, B, C respectively, with respect to some origin O. Then the position vectors of the mid-points D, E, F of BC, CA, AB respectively can be immediately computed.

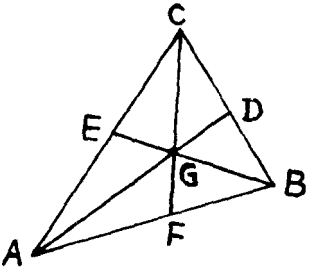


Fig 8.26

Denoting them by $\bar{d}$, $\bar{e}$, $\bar{f}$ respectively

we have $\bar{d} = \dfrac{\bar{b}+\bar{c}}{2}$, $\bar{e} = \dfrac{\bar{c}+\bar{a}}{2}$, $\bar{f} = \dfrac{\bar{a}+\bar{b}}{2}$.

Method 1 : We have

$$2\bar{d} = \bar{b} + \bar{c} \quad \ldots \text{(i)}, \qquad 2\bar{e} = \bar{c} + \bar{a} \quad \ldots \text{(ii)}$$

and $2\bar{f} = \bar{a} + \bar{b}$. $\qquad \ldots \text{(iii)}$

(i) − (ii) gives, $2\bar{d} - 2\bar{e} = \bar{b} - \bar{a}$, i. e., $\bar{a} + 2\bar{d} = \bar{b} + 2\bar{e}$.

$$\therefore \quad \frac{\bar{a} + 2\bar{d}}{1+2} = \frac{\bar{b} + 2\bar{e}}{1+2}. \qquad \ldots \ldots \text{(iv)}$$

Similarly, (i) − (iii) gives

$$2\bar{d} - 2\bar{f} = \bar{c} - \bar{a}, \text{ i. e., } \bar{c} + 2\bar{f} = \bar{a} + 2\bar{d}.$$

$$\therefore \quad \frac{\bar{a} + 2\bar{d}}{1+2} = \frac{\bar{c} + 2\bar{f}}{1+2}. \qquad \ldots \ldots \text{(v)}$$

From (iv) and (v), we get

$$\frac{\bar{a} + 2\bar{d}}{1+2} = \frac{\bar{b} + 2\bar{e}}{1+2} = \frac{\bar{c} + 2\bar{f}}{1+2} = \bar{g}, \text{ say.}$$

Hence AD, BE, CF meet in a point G with p. v. $\bar{g}$.

Further $\bar{g} = \dfrac{\bar{a} + 2\bar{d}}{1+2} = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3}$.

$\therefore$ AD, BE, CF meet in a point G with p. v. $\bar{g}$ which divides each one of them in the ratio 2 : 1.

Method 2 : Let G_1, G_2, G_3, with p. v. s. $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, divide $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ in the ratios $\alpha : 1$, $\beta : 1$, $\gamma : 1$ respectively.

$$\therefore \quad \bar{g}_1 = \frac{1\cdot\bar{a} + \alpha\left(\dfrac{\bar{b}+\bar{c}}{2}\right)}{1+\alpha}, \qquad \bar{g}_2 = \frac{1\cdot\bar{b} + \beta\left(\dfrac{\bar{c}+\bar{a}}{2}\right)}{1+\beta},$$

and $$\bar{g}_3 = \frac{1\cdot\bar{c} + \gamma\left(\dfrac{\bar{a}+\bar{b}}{2}\right)}{1+\gamma}.$$

If we choose $\alpha = \beta = \gamma = 2$, then

$\bar{g}_1 = \bar{g}_2 = \bar{g}_3 = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3} = \bar{g}$, say. Hence the result.

Ex. 10 : Prove that the internal bisectors of the angles of a triangle are concurrent.

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the p. v. s. of the vertices A, B, C respectively of a $\triangle ABC$.

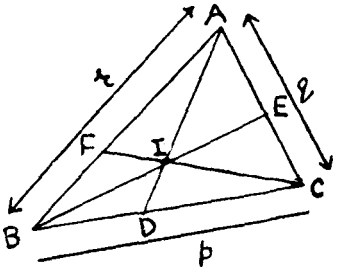


Fig. 8.27

Let seg AD, seg BE and seg CF be the internal bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively.

Let $BC = p$, $CA = q$ and $AB = r$.

We know from Geometry that

$$\frac{BD}{DC} = \frac{BA}{AC} = \frac{r}{q} .$$

$$\therefore \frac{BD}{BD + DC} = \frac{r}{q + r} .$$

$$\therefore BD = \frac{pr}{q + r} .$$

Let I_1 be a point on AD such that $\dfrac{AI_1}{I_1D} = \dfrac{q + r}{p}$.

If $\bar{d}$ is the position vector of $\bar{D}$ then by the section formula we have

$$\bar{d} = \frac{q\bar{b} + r\bar{c}}{q + r} .$$

Hence if $\bar{i}_1$ is the position vector of I_1, then since $\dfrac{AI_1}{I_1D} = \dfrac{q + r}{p}$, we have by the section formula

$$\bar{i}_1 = \frac{p\bar{a} + (q + r)\bar{d}}{p + q + r} = \frac{p\bar{a} + (q + r)\left(\dfrac{q\bar{b} + r\bar{c}}{q + r}\right)}{p + q + r} .$$

$$\therefore \bar{i}_1 = \frac{p\bar{a} + q\bar{b} + r\bar{c}}{p + q + r} . \qquad \ldots \text{(i)}$$

Similarly if I_2, with p. v. $\bar{i}_2$, divides seg BE in the ratio $(r + p)/q$, then we will have

$$\bar{i}_2 = \frac{q\bar{b} + r\bar{c} + p\bar{a}}{q + r + p} . \qquad \ldots \text{(ii)}$$

Similarly if I_3, with p.v. $\bar{i}_3$, divides seg CF in the ratio $(p + q)/r$, then we will get

$$\bar{i}_3 = \frac{r\bar{c} + p\bar{a} + q\bar{b}}{r + p + q} . \qquad \ldots \text{(iii)}$$

Since $\bar{i}_1 = \bar{i}_2 = \bar{i}_3 =$, say $\bar{i}$, therefore, the internal bisectors AD, BE, CF are concurrent at I whose position vector is given by

$$\bar{i} = \frac{p\bar{a} + q\bar{b} + r\bar{c}}{p + q + r} .$$

Ex. 11. If G is the centroid of a $\triangle ABC$, find the position vector of G and show that $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$. Further if G_1 is the centroid of another $\triangle A_1B_1C_1$ show that $\overline{AA_1} + \overline{BB_1} + \overline{CC_1} = 3\,\overline{GG_1}$.

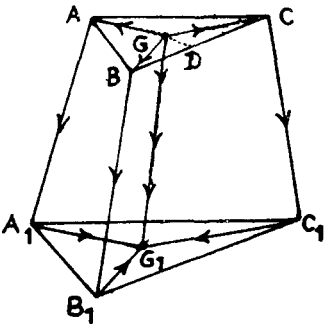


Fig. 8.28

Let AD be a median of $\triangle ABC$. Let the centroid G be taken as the origin.

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ be the position vectors of A, B, C and D respectively so that $\bar{d} = \frac{1}{2}(\bar{b} + \bar{c})$.

$\because$ G divides seg AD in the ratio 2 : 1,

$\therefore$ the position vector of G, say, $\bar{g}$ is given by

$$\bar{g} = \frac{1 \cdot \bar{a} + 2\left(\frac{\bar{b} + \bar{c}}{2}\right)}{1 + 2}.$$

But G, being the origin, $\bar{g} = 0$.

$$\therefore \quad \bar{a} + \bar{b} + \bar{c} = \bar{0}, \text{ i. e., } \overline{GA} + \overline{GB} + \overline{GC} = \bar{0}. \qquad \ldots \text{ (i)}$$

Now let $\bar{a_1}$, $\bar{b_1}$, $\bar{c_1}$ and $\bar{g_1}$ be the position vectors of A_1, B_1, C_1 and G_1 respectively w. r. t. the same point G as the origin.

$\therefore$ by the Polygonal Law of addition, we have

$$\overline{GA} + \overline{AA_1} + \overline{A_1G_1} = \overline{GG_1}. \qquad \ldots \text{ (i)}$$

Similarly, $\overline{GB} + \overline{BB_1} + \overline{B_1G_1} = \overline{GG_1}$, $\qquad \ldots$ (ii)

and $\overline{GC} + \overline{CC_1} + \overline{C_1G_1} = \overline{GG_1}$. $\qquad \ldots$ (iii)

By adding (i), (ii) and (iii), we get

$$(\overline{GA} + \overline{GB} + \overline{GC}) + (\overline{A_1G_1} + \overline{B_1G_1} + \overline{C_1G_1}) + (\overline{AA_1} + \overline{BB_1} + \overline{CC_1}) = 3\overline{GG_1},$$

i. e. $(\overline{GA} + \overline{GB} + \overline{GC}) - (\overline{G_1A_1} + \overline{G_1B_1} + \overline{G_1C_1}) + (\overline{AA_1} + \overline{BB_1} + \overline{CC_1}) = 3\overline{GG_1}$.

By using result (i) obtained above, we get

$\bar{0} - \bar{0} + \overline{AA_1} + \overline{BB_1} + \overline{CC_1} = 3\,\overline{GG_1}$. Hence the result.

Ex. 12. Show that the segment joining the mid-points of the non-parallel sides of a trapezium is parallel to the parallel sides and is one-half the sum of their lengths.

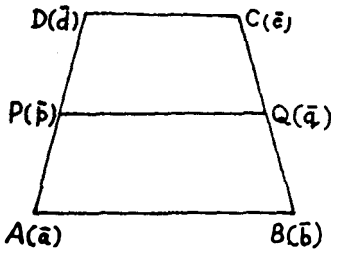


Fig. 8·29

Let $ABCD$ be a trapezium in which $AB \parallel DC$ and let P and Q be the mid-points of AD and BC.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}$ and $\bar{q}$ be the position vectors of A, B, C, D, P and Q respectively.

Then $\bar{p} = \dfrac{\bar{a} + \bar{d}}{2}$ and $\bar{q} = \dfrac{\bar{b} + \bar{c}}{2}$.

$$\therefore \overline{PQ} = \bar{q} - \bar{p} = \frac{\bar{b} + \bar{c}}{2} - \frac{\bar{a} + \bar{d}}{2} = \frac{\bar{b} - \bar{a}}{2} + \frac{\bar{c} - \bar{d}}{2} = \frac{\overline{AB} + \overline{DC}}{2}.$$

Now since $\overline{DC} \parallel \overline{AB}$, therefore, $\overline{DC} = k\overline{AB}$ and $DC = kAB$ where k is a constant.

$$\therefore \overline{PQ} = \frac{(k+1)}{2}\overline{AB}.$$

$\therefore PQ \parallel AB \parallel DC$ and $PQ = \frac{1}{2}(AB + DC)$.

Ex. 13. Show that the segment joining the mid-points of the diagonals of a trapezium is parallel to its parallel sides and is one-half the difference between their lengths.

Let P, Q be the mid-points of the diagonals BD and AC of the trapezium $ABCD$ in which $AB \parallel DC$.

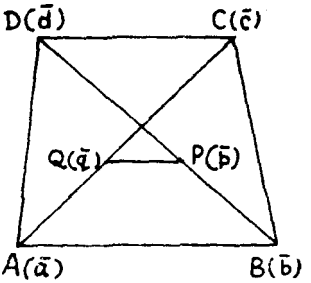


Fig. 8·30

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}, \bar{q}$ be the position vectors of A, B, C, D, P and Q respectively.

Then $\bar{p} = \dfrac{\bar{b} + \bar{d}}{2}$ and $\bar{q} = \dfrac{\bar{a} + \bar{c}}{2}$.

$$\therefore \overline{PQ} = \bar{q} - \bar{p} = \frac{\bar{a} + \bar{c}}{2} - \frac{\bar{b} + \bar{d}}{2}$$

$$= \frac{\bar{c} - \bar{d}}{2} - \frac{\bar{b} - \bar{a}}{2} = \frac{\overline{DC}}{2} - \frac{\overline{AB}}{2}. \quad \ldots (i)$$

Also since $DC \parallel AB$,

$\therefore \overline{DC} = k\overline{AB}$ and $DC = kAB$ where k is a constant.

$\therefore$ from (i), $\overline{PQ} = \dfrac{(k-1)}{2}\overline{AB}$.

$$\therefore PQ \parallel AB \parallel DC \text{ and } PQ = \left| \frac{kAB}{2} - \frac{AB}{2} \right| = \frac{1}{2}\left| (DC - AB) \right|.$$

Ex. 14. ABC is a triangle and D, E are the mid-points of the sides AB, AC respectively. Resolve the vector $\overline{DE}$ into its components along $\overline{AB}$ and $\overline{AC}$.

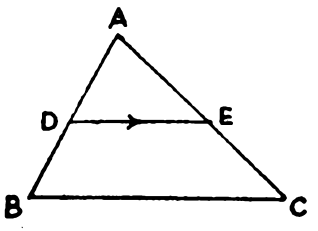


Fig. 8.31

Choosing A as the origin, let $\bar{b}$ and $\bar{c}$ be the position vectors of B and C. Then the position vectors of D, E are $\frac{\bar{b}}{2}$ and $\frac{\bar{c}}{2}$.

Hence, $\overline{DE} = \frac{\bar{c}}{2} - \frac{\bar{b}}{2}$. Since this is already a linear combination of $\overline{AB}$ $(= \bar{b})$ and $\overline{AC}$ $(= \bar{c})$, we have

$\overline{DE} = \left(\frac{1}{2}\overline{AC}\right) + \left(-\frac{1}{2}\right)\overline{AB}$ and so the components of $\overline{DE}$ along $\overline{AB}$, $\overline{AC}$ are $-\frac{1}{2}\overline{AB}$ and $\frac{1}{2}\overline{AC}$ respectively.

With the choice of $\overline{AB}$, $\overline{AC}$ as the two vectors along which a vector is to be resolved, the vector $\overline{DE}$ corresponds to the ordered pair $(-1/2, 1/2)$ which may be written as the row vector $[-1/2, 1/2]$, or as a column vector $\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$.

Alternatively we have

$$\overline{DE} = \overline{DA} + \overline{AE} = -\overline{AD} + \overline{AE}$$
$$= -\frac{1}{2}\overline{AB} + \frac{1}{2}\overline{AC}.$$

$\therefore$ the desired ordered pair is $(-1/2, 1/2)$.

Ex. 15. Find the ordered pair corresponding to $\overline{DE}$ in example 14 above when $\overline{DE}$ is resolved along $\overline{BD}$ and $\overline{AE}$.

We have $\overline{DE} = \overline{DA} + \overline{AE} = \overline{AE} + \overline{BD} = \overline{BD} + \overline{AE}$ and so the desired ordered pair is $(1, 1)$ which is different from the ordered pair $(-1/2, 1/2)$ obtained in example 14.

EXERCISE 8 (b)

(1) Prove the triangle inequality for vectors : For any two vectors $\bar{u}$, $\bar{v}$ $|\bar{u} + \bar{v}| \leqslant |\bar{u}| + |\bar{v}|$. State when the equality in the result holds.

(2) Prove the extended triangle inequality : For any vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$, $|\bar{u}_1 + \bar{u}_2 + \ldots\ldots + \bar{u}_n| \leqslant |\bar{u}_1| + |\bar{u}_2| + \ldots\ldots + |\bar{u}_n|$. State when the equality sign in the result holds.

(3) Prove the reverse triangle inequality : For any vectors $\bar{u}, \bar{v}$, $|\bar{u} - \bar{v}| \geq |\,|\bar{u}| - |\bar{v}|\,|$. State when the equality sign in the result holds.

(4) Prove that $\lambda\overline{OA} + \mu\overline{OB} = (\lambda + \mu)\,\overline{OD}$ where O, A, B are non-collinear and D is a point on AB such that $\lambda\,\overline{AD} = \mu\,\overline{DB}$.

(5) D, E, F are the mid-points of the sides BC, CA, AB respectively of a ΔABC and O is any other point. Prove that (i) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OD} + \overline{OE} + \overline{OF}$, (ii) $\overline{AD} + \overline{BE} + \overline{CF} = \bar{0}$.

(6) H is the orthocentre and O is the circumcentre of a ΔABC. Show that

(i) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$,

(ii) $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{AO} + 2\overline{BO} + 2\overline{CO}$.

(7) O is any point in the plane of a ΔABC; prove that

$p\,\overline{BC} + q\,\overline{CA} + r\,\overline{AB} = (r - q)\,\overline{AO} + (p - r)\,\overline{BO} + (q - p)\,\overline{CO}$, where p, q, r are scalars.

(8) G is a point in the plane of a ΔABC such that $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$; show that G is the centroid of the ΔABC.

(9) Show that in a triangle, the segment joining the mid-points of two sides is parallel to the third side and is one half this side in length.

(10) $ABCD$ is a quadrilateral. If M and N are the mid-points of AB and CD respectively, show that

$$\overline{AD} + \overline{BC} = 2\overline{MN}.$$

(11) $ABCDEF$ is a regular hexagon, prove that

(i) $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD}$.

(ii) $\overline{AB} + \overline{AF} + \overline{BC} + \overline{FE} + \overline{CD} + \overline{DE} = 2\overline{AE}$.

(12) Prove that the points with position vectors $\bar{p}, \bar{q}, \bar{r}$ are collinear if

(i) $\bar{p} = \bar{a} - 2\bar{b} + 3\bar{c}$, $\bar{q} = -2\bar{a} + 3\bar{b} - 2\bar{c}$, $\bar{r} = 7\bar{a} - 12\bar{b} + 13\bar{c}$,

(ii) $\bar{p} = \bar{a} - 2\bar{b} + 3\bar{c}$, $\bar{q} = 2\bar{a} + 3\bar{b} - 4\bar{c}$, $\bar{r} = -7\bar{b} + 10\bar{c}$.

(iii) $\bar{p} = (1, -2)$, $\bar{q} = (2, -4)$, $\bar{r} = (-1, 2)$.

(13) Prove that the vectors $\overline{PQ}$ and $\overline{RS}$ are parallel, if the position vectors of P, Q, R, S are given by

$\bar{p} = 3\bar{a} + 5\bar{b} - 2\bar{c}$, $\bar{q} = \bar{a} + \bar{b} - 2\bar{c}$, $\bar{r} = 4\bar{a} + 3\bar{b} - 3\bar{c}$,

$\bar{s} = 3\bar{a} + \bar{b} - 3\bar{c}$.

(14) $\bar{a}, \bar{b}, \bar{c}$ are the position vectors of three points A, B, C respectively. If the point P divides seg AB internally in the ratio $3 : 2$ and if Q is the mid-point of seg BC, express $\overline{PQ}$ in terms of $\bar{a}, \bar{b}, \bar{c}$.

(15) In a quadrilateral $ABCD$, P, Q, R, S are the mid-points of the sides AB, BC, CD and DA respectively; show that $PQRS$ is a parallelogram.

(16) AB and CD are two chords of a circle intersecting each other at right angles at P. If O is the centre of the circle, prove that

$$\overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} = 2\overline{PO}.$$

(17) In a triangle ABC, P is the mid-point of BC and Q and R lie on CA and AB respectively such that

$\frac{CQ}{QA} = 2$ and $\frac{AR}{RB} = -\frac{1}{2}$, show that P, Q, R are collinear.

(18) E and F are the mid-points of the sides AB and CD of a quadrilateral $ABCD$. Resolve the vector EF along the diagonals AC and BD.

(19) If $\bar{a} = 2\bar{i} - 3\bar{j}$, $\bar{b} = \bar{i} + 2\bar{j}$ are two given vectors in a plane, find the vector $\bar{c} = 9\bar{i} - 4\bar{j}$ in terms of $\bar{a}$ and $\bar{b}$.

(20) If the vector $\overline{OP} = (2, 5)$, find the magnitude and direction of $\overline{OP}$.

(21) Find the angle between the two vectors $\overline{OP}$ and $\overline{OQ}$, where $\bar{p} = (2, 3)$ and $\bar{q} = (4, 5)$.

(22) If $\bar{a} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$, $\bar{d} = \begin{pmatrix} 12 \\ 27 \end{pmatrix}$ and $\bar{d} = x\bar{a} + y\bar{b}$, find x and y.

(23) (i) If O is the origin and $\overline{OA} = (1, 2)$, $\overline{OB} = (4, 6)$ and $\overline{OC} = (-3, -1)$, then show that A is equidistant from B and C.

(ii) If O is the origin and $\overline{OA} = (2, 2)$, $\overline{OB} = (5, 6)$, $\overline{OC} = (1, 9)$ and $\overline{OD} = (-2, 5)$, then show that $ABCD$ is a square.

(iii) If O is the origin and $\overline{OA} = (2, -5)$, $\overline{OB} = (4, 3)$, $\overline{OC} = (3, 6)$ and $\overline{OD} = (2, 2)$, show that $\overline{AB}$ and $\overline{CD}$ are parallel.

(24) If $\bar{a} = 4\bar{i} - 5\bar{j}$, $\bar{b} = 8\bar{i} + \bar{j}$, $\bar{c} = 14\bar{i} - 3\bar{j}$ are the position vectors of the vertices A, B, C of a $\triangle ABC$, show that $\triangle ABC$ is a right-angled.

(25) If $\bar{a}$ and $\bar{b}$ are two non-zero, non-parallel vectors such that

$$(x + y)\bar{a} + (x - y)\bar{b} = 3\bar{a}, \text{ find } x \text{ and } y.$$

(26) If $\bar{a}$ and $\bar{b}$ are two non-collinear vectors and $3\bar{p} = 2\bar{q}$, where

$$\bar{p} = (x + 4y)\bar{a} + (2x + y + 1)\bar{b}$$

and $\bar{q} = (y - 2x + 2)\bar{a} + (2x - 3y - 1)\bar{b}$, find x and y.

(27) If $\bar{r}_1 = -\bar{i} + 2\bar{j}$, $\bar{r}_2 = 5\bar{i} - 3\bar{j}$, $\bar{r}_3 = 4\bar{i} - \bar{j}$,

(a) find the magnitudes of the vectors

(i) $\bar{r}_1 + \bar{r}_2 + \bar{r}_3$, (ii) $2\bar{r}_1 - \bar{r}_2 + 3\bar{r}_3$;

(b) find the unit vector parallel to the resultant of $\bar{r}_1$ and $\bar{r}_2$.

(28) **Fill in the blanks in the following :—**

(i) If O, A, B are three distinct points, then $\overline{OA} - \overline{OB} =$ .....

(ii) If A, B, C are three collinear points with position vectors $\bar{a}, \bar{b}, \bar{c}$ respectively and $k\bar{a} + 2\bar{b} - 3\bar{c} = \bar{0}$, then $k =$ .....

(iii) If A, B, C are three points with p. v. s $\bar{a}, \bar{b}, \bar{c}$ respectively and $7\bar{a} - 4\bar{b} - 3\bar{c} = \bar{0}$, then A, B, C are ..... points.

(iv) If $\overline{AO} + \overline{OB} = \overline{BO} + \overline{OC}$ then A, B, C are ..... points.

(v) If $\overline{BX} + \overline{AB} = \overline{AX} + \overline{YX}$, then X and Y are ..... points.

(vi) If L, M, N are the mid-points of the sides AB, BC, CA respectively of a $\triangle ABC$, then $\overline{AL} + \overline{BM} + \overline{CN} =$ .....

(vii) PQ is a fixed diameter of a circle. If A be any point on this circle such that $\overline{AP} + \overline{AQ} = \overline{AB}$, then the locus of B is .....

(viii) The position vectors of the vertices A, B, C of a parallelogram $ABCD$ are $2\bar{i} + 5\bar{j}$, $\bar{i} + 2\bar{j}$, $3\bar{i} - 2\bar{j}$ respectively. Hence the p. v. of D is .....

(ix) G is the centroid of a $\triangle ABC$ such that
$\bar{g} = 3\bar{i} + 7\bar{j}$, $\bar{a} = 2\bar{i} + \bar{j}$, $\bar{b} = x\bar{i} + 3\bar{j}$, $\bar{c} = 4\bar{i} + y\bar{j}$.
Hence $x =$ ....., $y =$ .....

(x) The position vectors of the points A, B, P are $2\bar{i} + 3\bar{j}$, $6\bar{i} + 9\bar{j}$ and $4\bar{i} + 6\bar{j}$ respectively. Hence $\overline{AB} =$ ..... $\overline{AP}$ and the points A, P, B are .....

(29) State whether the following statements are true or false.

(i) For any point D on the side BC of a $\triangle ABC$,
$\overline{AB} + \overline{BC} = \overline{AD} + \overline{DC}$.

(ii) For any three points A, B, C if $\bar{a} = \overline{BC}$, $\bar{b} = \overline{CA}$ and $\bar{c} = \overline{AB}$, then $\bar{a} = -\bar{b} - \bar{c}$ and $\bar{a} + 2\bar{b} = \bar{b} - \bar{c}$.

(iii) In a regular hexagon $ABCDEF$, if $\bar{a} = \overline{AB}$ and $\bar{b} = \overline{AF}$, then $\bar{a} + \bar{b} = \overline{BC}$.

(iv) The points with position vectors, given by the ordered pairs, $(1, -2)$, $(2, -4)$, $(-1, 2)$ are collinear.

(v) In a quadrilateral $ABCD$ a point O is such that $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = \bar{0}$. Hence O is the point of intersection of its diagonals.

(vi) If $a = b$, then $\bar{a} = \bar{b}$.

(vii) If $\bar{a} = \bar{b}$, then $a = b$.

(30) Choose the correct answer from the given alternatives in the following.

(i) If $ABCDEF$ is a regular hexagon with its centre at O, then

(a) $\overline{AB} = \overline{DE}$, (b) $\overline{AB} + \overline{BC} + \overline{CD} = 2\overline{AO}$,

(c) $\overline{AB} + \overline{BC} + \overline{CD} = 2\overline{EF}$, (d) $\overline{AD} = 3\overline{BC}$.

(ii) $\overline{AP} = 3\overline{PB}$, then P divides seg AB in the ratio

(a) 1/3, (b) 3/1, (c) 1/4, (d) none of these.

(iii) If no three of the four points A, B, C, D with position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are collinear and if $\bar{a} + \bar{c} = \bar{b} + \bar{d}$ then $ABCD$ is always

(a) a parallelogram, (b) a rectangle, (c) a rhombus.

(iv) If the ordered pairs (1, − 2) and (5, − 4) are the position vectors of the points A and B then the unit vector in the direction of $\overline{AB}$ is

(a) $\frac{-2\bar{i} + \bar{j}}{\sqrt{5}}$, (b) $\frac{4\bar{j} - 2\bar{i}}{2\sqrt{5}}$, (c) $-4i + 2j$, (d) none of these.

(v) If $\overline{AP} = 4\overline{BP}$, then P divides seg AB in the ratio

(a) 1/4, (b) 4/1, (c) 1/5, (d) none of these.

Hints and Answers :

(1) Equality holds when $\bar{u}$, $\bar{v}$ have the same direction.

(2) Equality holds when $\bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots$ all have the same direction.

(3) Equality holds when $\bar{u}$, $\bar{v}$ have the same direction.

(4) **Hint** : $\lambda\overline{OA} = \lambda\overline{OD} + \lambda\overline{DA}$, $\mu\overline{OB} = \mu\overline{OD} + \mu\overline{DB}$ and $\lambda\overline{DA} + \mu\overline{DB} = \bar{0}$.

(5) **Hint** : (i) $\overline{OB} + \overline{OC} = 2\overline{OD}$ etc.

(ii) $\overline{AD} = \frac{3}{2}\overline{AG}$ etc. where G is the centroid of $\triangle ABC$ and $\overline{AG} + \overline{BG} + \overline{CG} = \bar{0}$.

(6) See Fig. 8.32. **Hint** : (i) Draw seg OL perpendicular to BC and produce OL to L' so that seg LL' = seg OL. Use the property that $AH = 2\,OL$ and hence show that $AOL'H$ is a parallelogram. (ii) $\overline{AH} = 2\overline{OL} = \overline{OB} + \overline{OC}$ etc.

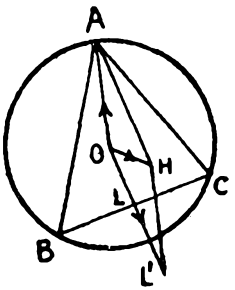


Fig. 8.32

(8) **Hint** : Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of A, B, C respectively with respect to G as the origin. Then $\bar{a} + \bar{b} + \bar{c} = \bar{0}$ gives

$$\bar{0} = \bar{g} = \frac{1\cdot\bar{a} + 2\cdot\left(\frac{\bar{b} + \bar{c}}{2}\right)}{1 + 2}.$$

Hence G divides the median AD in the ratio 2:1.

Similarly for the other medians.

(12) **Hint :** (i) $\bar{r} - \bar{p} = 2(\bar{p} - \bar{q})$; (ii) $(\bar{q} - \bar{p}) = (\bar{p} - \bar{r})$;
(iii) $\bar{r} - \bar{p} = 2(\bar{p} - \bar{q})$.

(13) **Hint :** $\overline{PQ} = 2\overline{RS}$. (14) $\dfrac{5\bar{c} - 4\bar{a} - \bar{b}}{10}$.

(16) Seg Fig. 8.33. **Hint :** Draw seg OR perpendicular to AB and seg OS perpendicular to CD, If $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{r}, \bar{s}$ are the position vectors of A, B, C, D, R and S respectively, with respect to P as the origin, then

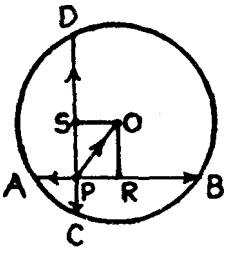


Fig. 8·33

$2\bar{r} = \bar{a} + \bar{b}$, $2\bar{s} = \bar{c} + \bar{d}$ and $\overline{PO} = \bar{r} + \bar{s}$.

(17) **Hint :** If $\bar{a}, \bar{b}, \bar{c}, \bar{p}, \bar{q}, \bar{r}$ be the position vectors of A, B, C, P, Q, R respectively then

$$\bar{p} = \frac{\bar{b} + \bar{c}}{2}, \bar{q} = \frac{2\bar{a} + \bar{c}}{3}, \bar{r} = 2\bar{a} - \bar{b} \text{ and}$$

hence $\bar{r} = \dfrac{3\bar{q} - 2\bar{p}}{3-2}$, i. e., R divides seg PQ in the ratio $3 : -2$.

(18) $\frac{1}{4}\overline{AC}$, $\frac{1}{3}\overline{BD}$. (19) $\bar{c} = \dfrac{22\bar{a} + 19\bar{b}}{7}$.

(20) $|\overline{OP}| = \sqrt{29}$; $\overline{OP}$ makes an angle $\cos^{-1}\left(\dfrac{2}{\sqrt{29}}\right)$ with the x-axis. (21) $\cos^{-1}\left(\dfrac{23}{\sqrt{533}}\right)$.

(22) $x = -6$, $y = 21$.

(24) **Hint :** $|\bar{b} - \bar{a}|^2 + |\bar{b} - \bar{c}|^2 = |\bar{a} - \bar{c}|^2$.

(25) $x = y = \dfrac{3}{2}$. (26) $x = 2$, $y = -1$.

(27) (a) (i) $2\sqrt{17}$, (ii) $\sqrt{41}$, (b $\left(\dfrac{4\bar{i} - \bar{j}}{\sqrt{17}}\right)$.

(28) (i) $\overline{BA}$, (ii) 1, (iii) collinear, (iv) collinear, (v) coincident, (vi) $\bar{0}$, (vii) the same circle, (viii) $4\bar{i} + \bar{j}$, (ix) 3, 17, (x) 2, collinear.

(29) (i) True, (ii) True, True, (iii) True, (iv) True, (v) False; true only if $ABCD$ is a parallelogram. (vi) False. (vii) True.

(30) (i) b, (ii) b, (iii) a, (iv) d, (v) d.

□ □ □

CHAPTER 9

Prisms and Cylinders

(1) **Introduction**: You have been already introduced to the subject of Pure Solid Geometry earlier. You have also gathered a fairly broad idea of the concepts of the indefinables in Geometry, viz. points, lines, planes and surfaces. You have been also informally introduced to certain solids, viz the tetrahedron, the parallelopiped, prisms, pyramids, cones, cylinders and the sphere, through appropriate diagrams. Your study was based on certain postulates and definitions and you have also proved certain theorems on the basis of these.

Our purpose now is to study in some detail and deal with the problem of mensuration—calculation of surface areas and volumes—of the above solids.

However, before we start consideration of these problems, it would be advisable to refresh your memory by stating some of the important results that you know already.

You know, amongst others, the following results on the mutual relationships between lines and planes:

Review

(i) If lines are perpendicular to one and the same plane, they are parallel to one another.

(ii) If one of two parallel lines be perpendicular to a given plane, then so is the other.

(iii) If a line be perpendicular to one of two parallel planes then it is also perpendicular to the other plane.

(iv) The common sections of two parallel planes by another intersecting plane are lines parallel to each other as well as the generalisation of this result for a series of parallel planes.

(v) Dihedral Angle. You also have the concept of the dihedral angle of two intersecting planes. Thus let the half planes α and β intersect in the line XY (See Fig. 9.1). Take any point O in XY. At O draw two lines, say, OA in the half plane α and OB in the half plane β, both being perpendicular to XY. Then $m(\widehat{AOB})$ is defined to be the

dihedral angle of the two half planes α and β. Its measure does not depend on the choice of a point O in the line XY. Note also that the planes α and β are half planes bounded by the line XY. We shall briefly denote this dihedral angle by di. $\widehat{\alpha,\beta}$.

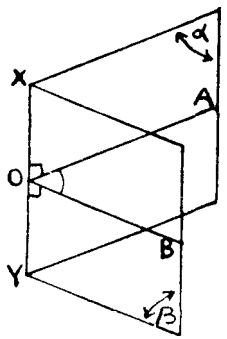


Fig. 9·1

We cannot obviously revise the whole of Solid Geometry you have studied. Hence, you are now advised to revise all the theorems, which have not been mentioned above, in Solid Geometry, that you have studied so far in your school.

(2) Polyhedron :

Definition : *A solid figure bounded by planes is called a Polyhedron.* The bounding planes are called the **faces**, the intersections of adjoining faces are called the **edges** and the intersections of adjoining edges are called the **vertices** of the polyhedron. A line segment joining two vertices of a ploy hedron, not lying in any of its faces, is called **a diagonal** of the polyhedron.

A polyhedron is named in accordance with the number of faces by which it is bounded.

Definition : *A polyhedron of* ($n + 2$) *faces, two of which, called the bases, are congruent polygons* (*with corresponding sides parallel*) *of n sides, lying in parallel planes and in which the remaining n faces, called the lateral faces, are parallelograms, is called a* **prism.**

Definition : *A polyhedron of* ($n + 1$) *faces, one of which is a polygon, called the base, of n sides and in which the remaining n faces, called the lateral faces, are triangles having a common vertex which does not lie in the plane of the base polygon of n sides, is called a* **pyramid.**

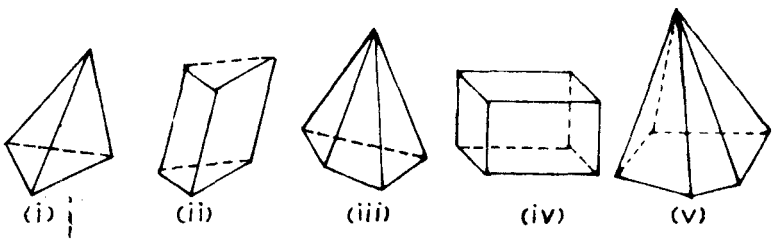


Fig. 9·2

In particular, a polyhedron with four faces [Fig. 9·2 (i)] is called a tetrahedron, one with five faces [Fig. 9·2 (ii) and 9·2 (iii)] a pentahedron, one with six faces [Fig. 9·2 (iv) and Fig. 9·2 (v)] a hexahedron. Figures (i), (iii), (v) are pyramids whereas figures (ii) and (iv) are prisms.

A prism is named in accordance with the number of sides of its base polygons.

Thus, (i) a prism having triangular bases is called a **triangular prism.**

(ii) a prism having parallelogram bases is called a **parallelopiped.**

(iii) a prism having pentagonal or a hexagonal bases is called a **pentagonal** or a **hexagonal** prism respectively.

In general, a prism having n-gonal bases is called an n-gonal prism.

Similarly a pyramid having triangular, pentagonal, hexagonal or in general n-gonal base is called a triangular pyramid (tetrahedron). pentagonal, hexgonal or in general n-gonal pyramid.

We propose to consider in this chapter the problem of mensuration of a prism, and its limiting form the cylinder, and postpone the discussion of the mensuration of a pyramid, and its limiting form the cone, to the next chapter. Before we consider prisms, in general, we first prove some results in the case of a special kind of a prism viz. a parallelopiped (which is a special kind of a hexahedron) with which students are generally more familiar.

(3) **Parallelopiped**

Definition : *A Parallelopiped is a closed solid bounded by three pairs of parallel planes.*

Notation :

Consider three pairs of parallel planes : α, α'; β, β'; γ, γ'. (Fig. 9·3)

In the figure, there are *eight* vertices, *six* faces and 12 *edges.*

α face is the left-hand face $OBA'C$.

α' .... right-hand .... $AC'VB'$.

β .... front .... $OAB'C$.

β' .... back .... $BC'VA'$.

γ .... bottom .... $OAC'B$.

γ' .... top .... $CB'VA'$.

Theorem. Opposite faces of a parallelopiped are congruent parallelograms.

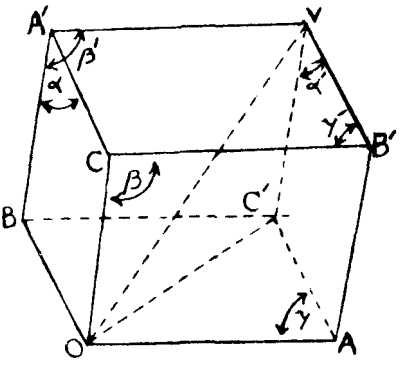


Fig. 9·3

Proof : In the figure 9·3, the intersections of the parallel planes α, α' by the plane γ are OB and AC'. $\therefore$ $OB \parallel AC'$.

Similarly $CA' \parallel B'V$.

Again, considering the intersections of the parallel planes γ and γ' by the cutting plane α', we get $AC' \parallel B'V$.

$\therefore$ $OB \parallel AC' \parallel B'V \parallel CA'$.

Similarly, we have $OA \parallel BC' \parallel A'V \parallel CB'$,

and $OC \parallel AB' \parallel C'V \parallel BA'$.

Thus in the plane α, we have

seg $OB \parallel$ seg CA' and seg $OC \parallel$ seg BA'.

Thus, the figure $OBA'C$ is a parallelogram. Similarly, all the other faces in the other five planes are also parallelograms. Thus every parallelopiped has 3 pairs of opposite faces which are all parallelograms.

In the pair of opposite faces α, α', we have $OB =$ and $\parallel AC'$, $BA' =$ and $\parallel C'V$, $A'C =$ and $\parallel VB'$, $CO =$ and $\parallel B'A$; $\therefore$ the parallel faces α, α' are congruent parallelograms. Similarly the other pairs of opposite faces of the parallelopiped are congruent parallelograms.

Choosing the parallel congruent bottom and top faces as bases, the parallelopied may be denoted by $[OAC'B - CB'VA']$.

A parallelopiped may be referred to symbolically by the letter $\mathfrak{P}$.

Some Terms :

We shall now introduce some terms associated with a parallelopiped.

(**a**) **Vertices.** The points O, A, B, C, V, A', B' and C' are called the vertices of the $\mathfrak{P}$.

(**b**) **Edges.** The intersections of concecutive faces of a parallelopiped are called its **edges**. They are twelve in number and can be grouped into three groups as follows :

(i) seg OA, seg BC', seg $A'V$ and seg CB' which are parallel to one another and are of the same length, say x.

(ii) seg OB, seg CA', seg $B'V$ and seg AC' which are parallel to one another and are of the same length, say y, and

(iii) seg OC, seg AB', seg $C'V$ and seg BA' which are also parallel to one another and are each of length, say z.

(**c**) **Co-terminal edges.** The faces α, β, γ intersect in O and the edges seg OA, seg OB and seg OC meeting in a vertex O are called **co-terminus edges** of the parallelopiped. Similarly the faces α', β', γ' meet in V and seg VA', seg VB' and seg VC' are another three co-terminus edges

meeting in the vertex V. The points O and V are called a pair of opposite vertices. Similarly A, A'; B, B'; C, C' are three other pairs of opposite vertices.

(d) **Diagonals.** seg OV, seg AA', seg BB' and seg CC' which join the pairs of opposite vertices are called the **diagonals** of the parallelopiped.

(4) **Particular forms of a ℙ.**

Definitions :

(i) **Rectangular Parallelopiped.** A parallelopiped, every face of which is a rectangle, is called **a rectangular parallelopiped** or a **cuboid.** Fig. 9·4 (i).

(ii) **Cube.** A parallelopiped, every face of which is a square, is called a **cube.** Fig. 9·4 (ii).

(iii) **Rhomboid.** A parallelopiped, every face of which is a rhombus is called a **rhomboid.** Fig. 9·4 (iii).

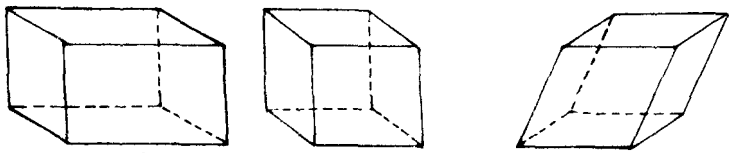


Fig. 9·4

(5) **Theorem : The four diagonals of a parallelopiped are concurrent and bisect one another.**

In Fig. 9·3, OC and $C'V$ being equal and parallel, $OCVC'$ is a parallelogram.

$\therefore$ the diagonals of this parallelogram viz. OV and CC' intersect and bisect each other.

Similarly OV and BB' are diagonals of the parallelogram $OBVB'$; hence they intersect and bisect each other.

Similarly OV and AA' are diagonals of the parallologram $OAVA'$; hence they intersect and bisect each other.

But a straight line, in particular here OV, has only one bisecting point.

$\therefore$ It follows that the four diagonals OV, AA', BB' and CC' meet in a point and bisect one another.

(6) Properties of a rectangular ℙ.

(i) Length of a diagonal of a rectangular ℙ.

Theorem : The square on a diagonal of a rectangular parallelopiped is equal to the sum of the squares on its three coterminus edges.

Let [$OAC'B - CB'VA'$] be a rectangular ℙ in which every face is a rectangle. Join OV and OC'. (see Fig. 9.3).

Now seg $C'V$ is $\perp$ to seg $C'A$ and seg $C'B$.

$\therefore$ seg $C'V \perp$ pl $OAC'B$. $\therefore$ seg $C'V \perp$ seg OC'.

$\therefore$ from the right-angled triangle $OC'V$, in which $m(\widehat{OC'V}) = 90°$, we have

$$OV^2 = (OC')^2 + (C'V)^2.$$

From $\triangle OAC'$ in which $m(\widehat{OAC'}) = 90°$, we have

$$(OC')^2 = OA^2 + (AC')^2.$$

Further $C'V = OC$ and $AC' = OB$;

$$\therefore \quad OV^2 = OA^2 + OB^2 + OC^2.$$

Thus if x, y, z be the lengths of the coterminous edges and d the length of a diagonal of a rectangular ℙ, we get

$$d^2 = x^2 + y^2 + z^2.$$

(ii) Volume and Surface Area

Remark : The measure of any area or volume depends on the unit chosen for measuring the same. For measuring an area, the unit chosen is the area of a unit square, i. e., a square whose side is one unit in length. It is on this basis that we have the formula "area of a rectangle is equal to the product of its adjacent sides ".

For measuring volumes, the unit chosen is the volume of a unit cube, i. e., a cube each of whose edges is one unit in length.

(a) Volume of a rectangular ℙ. The Volume of a rectangular ℙ whose base is a rectangle with adjacent sides x and y units in length and whose height is z units in lengh can now be determined.

The area of the base is xy square units.

The height of the solid being z units in length, we can arrange a pile of z unit cubes on each of the xy square units of the area of the base. Hence, the number of unit cubes contained in the solid is xyz.

$\therefore$ the volume of a rectangular ℙ = product of the lengths of its three coterminus edges $= xyz$.

Remark : The above discussion assumes that the sides of a rectangle and the edges of a ℙ are commensurable. If they are not commensurable, we can choose the unit of length sufficiently small so as to make the error as small as we please.

(b) Surface Area of a rectangular ℙ. The surface area of a rectangular ℙ being the sum of the areas of its six faces, which are all rectangles, congruent in pairs, will be

$$2xy + 2yz + 2zx.$$

We shall consider the general case of any ℙ in the next article.

Note : If the rectangular ℙ is a cube, then we have $x = y = z$ and hence (i) the length of each of its diagonals is $x\sqrt{3}$, (ii) its total surface area is $6x^2$ and (iii) its volume is x^3.

(7) Surface area of a general ℙ : If x, y, z are the lengths of three coterminus edges of a general ℙ and if $\theta = \widehat{x,y}$, $\phi = \widehat{y,z}$ and $\psi = \widehat{z,x}$, then its total surface area is $2(xy \sin\theta + yz \sin\phi + zx \sin\psi)$ since the area of a parallelogram with adjacent sides a, b is $ab \sin(\widehat{a,b})$.

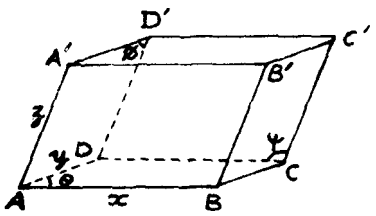


Fig. 9·5

Illustrative Examples

Ex. 1. The sum of the lengths of three coterminus edges of a cuboid is 1.21 meters and its diagonal is 29 cms. Find its total surface area.

Let a, b, c cms. be the lengths of the three coterminus edges and d cms. be the length of its diagonal.

Then $a + b + c = 121$ and $d = 29$.

$\therefore$ the total surface area

$= 2(bc + ca + ab) = (a + b + c)^2 - a^2 - b^2 - c^2$

$= (a + b + c)^2 - d^2 = (121)^2 - (29)^2$

$= 150 \times 92 = 13800$ sq. cms.

Ex. 2. The whole surface of a cuboid contains 724 sq. cms, the area of the base is 132 sq. cms. and one of the vertical faces contains 110 sq. cms. Determine the lengths of its edges and its volume.

Let a, b cms. be the edges of the base and c cms. be the vertical edge; then $ab = 132$, $bc = 110$, and $2(bc + ca + ab) = 724$.

$\therefore bc + ca + ab = 362$; $\therefore ac = 120$.

$\therefore a^2b^2c^2 = (ab)(bc)(ca) = (11 \times 12) \times (11 \times 10) \times (12 \times 10)$.

$\therefore abc = 10 \times 11 \times 12$.

$\therefore a = 12$ cms; $b = 11$ cms. and $c = 10$ cms.

$\therefore$ the volume of the cuboid $= abc = 12 \times 11 \times 10 = 1320$ cub. cms.

Ex. 3. Find the weight of a steel bar 1.28 meters long, 15 cms. wide and 5 cms. thick, the specific gravity of the bar being 7.8.

The volume of the steel bar

$= 128 \times 15 \times 5 = 9600$ cub. cms.

$\therefore$ the weight of the bar $= 9600 \times 7.8$ gms.

$= 9.6 \times 7.8$ kg. $= 74.88$ kg.

EXERCISE 9 (a)

(1) Two of the edges of a cuboid are 4 cms. and 6 cms. The volume of the cuboid is 192 c. cms. Find its total surface area and the length of its diagonal.

(2) [$OAC_1B - CB_1VA_1$] is a cuboid in which $OA = 12$ cms., $OB = 6$ cms. and $OC = 5$ cms. Find the areas of the rectangles OC_1VC and $OAVA_1$.

(3) Two of the edges of a cuboid are 7 cms. and 9 cms. and its total surface area is 446 sq. cms. Find its volume.

(4) The sum of the lengths of the edges of a cuboid is 60 cms. and its total surface area is 148 sq. cms. If its volume is 120 cub. cms. determine the lengths of its edges and also the length of a diagonal.

(5) The diagonals of the three faces of a cuboid meeting in a vertex of the cuboid are 5 cms., $3\sqrt{5}$ cms. and $2\sqrt{13}$ cms. Find its total surface and volume.

(6) A closed rectangular box is built of wood of uniform thickness. Its external dimensions are 12 cms., 10 cms., and 8 cms. and its inner surface is 376 sq. cms. Find the thickness of the wood.

(7) The whole surface of a rectangular block is 1332 sq. cms. Find its length, breadth and height, it being given that these dimensions are in the ratios 4 : 5 : 6.

(8) Find the longest rectangular rod that can be placed in a room of 10 m. × 8 m. × 9 m.

(9) The diagonal of a rectangular solid is 17 cms. and its whole surface area is 440 sq. cms. Find the sum of its three dimensions.

(10) The number of cub. meters contained in a cubical box is half the number of sq. meters in its surface. Find the volume of the box.

(11) Fill in the blanks in the following :

(i) The lengths of the coterminus edges of a rectangular parallelopiped are as 2 : 3 : 4. If its volume is 192 cub. cms., the lengths of the edges are ....., ..., ... cms

(ii) The volume of a cube is 343 cub. cms.

∴ its total surface is ... . sq. cms.

(iii) The lateral surface area of a rectangular box whose base is a square of side 9 m. is 216 sq. m. Hence its height = ... m. and its volume = ... cub. m.

(12) Answer the following.

V denotes the volume of a rectangular parallelopiped of height h and rectangular bases of sides a and b.

(a) If $a = 4$ cms., $b = 5$ cms. and $h = 6$ cms. its total surface area = ...sq. cms.

(b) If its volume is 3600 cub. cms., $a = 15$ cms. $h = 12$ cms. then $b =$ ... cms. and lateral surface = ... sq. cms,

(c) If $a = b$, lateral surface area = 600 sq. cms. and volume = 2250 cub. cms., then $a =$ ... cms. and $h =$ ...cms.

ANSWERS

(1) 208 sq. cms; $2\sqrt{29}$ cms. (2) $30\sqrt{5}$ sq. cms; $12\sqrt{61}$ sq. cms.
(3) 630 cub. cms. (4) 4 cms, 5 cms, 6 cms; $\sqrt{77}$ cms.
(5) 108 sq. cms; 72 c. cms. (6) 1 cm.
(7) 12 cms., 15 cms, 18 cms. (8) $7\sqrt{5}$ m.
(9) 27 cms. (10) 27 cub. m.
(11) (i) 4, 6, 8; (ii) 294, (iii) 6, 486.
(12) (a) 148, (b) 20, 840, (c) 15, 10.

(8) Prism. In article 2, we have defined a prism as a polyhedron bounded by $(n + 2)$ faces, two of which, called the bases, are congruent polygons of n sides (with corresponding sides parallel) lying in parallel planes, the remaining n faces, called the lateral faces, being all parallelograms. We now propose to state the following alternative definition.

Alternative Definition. If one end A of seg AA' is moved along the perimeter of a given polygon $ABC...K$, keeping the segment always parallel to a fixed direction in space, then its other end A' describes a polygon $A'B'C'...K'$, in a parallel plane, congruent with the given polygon $ABC...K$ and generates the lateral faces of a solid called a **prism.**

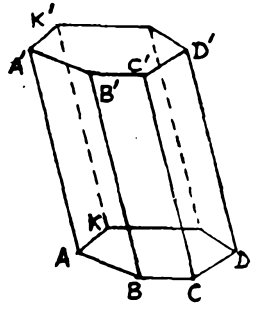


Fig. 9.6

Seg AA' in any of its positions is called a generator of the prism and the prism is denoted by $[ABC...K - A'B'C'...K']$.

Theorem. Lateral faces of a prism are all parallelograms.

AA' is equal and parallel to BB' (Fig. 9·6) $\therefore$ the lateral face $AA'B'B$ is a parallelogram. Similarly all the other lateral faces $BB'C'C$, $CC'D'D$...etc. are all parallelograms.

Definitions:

(i) The intersections of consecutive lateral faces such as AA', BB', CC' etc. are called the **lateral edges of the prism.**

Remark. Evidently the lateral edges of a prism are all equal and parallel to one another.

(ii) The **lateral area of a prism** is the sum of the areas of its lateral faces.

(iii) The **total area of a prism** is the sum of the areas of its lateral faces and its two congruent bases.

(iv) The **height** or **altitude** of a prism is the perpendicular distance between its congruent bases.

(v) A **diagonal** of a prism is a line segment, joining any two of its vertices, and not lying in any of its faces.

(9) Cylinder. Definition. If one end A of seg AA' is moved along

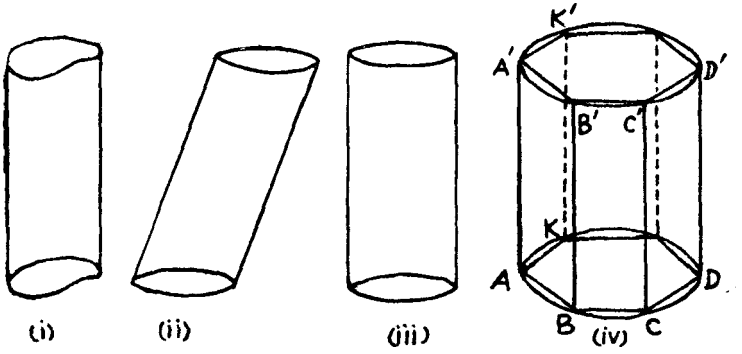


Fig. 9·7

the perimeter of a given closed plane curve ABC, keeping the segment always parallel to a fixed direction in space, then its other end A' describes a closed plane curve $A'B'C'$, in a parallel plane, congruent with the given closed plane curve ABC, and generates the lateral surface of a solid called a cylinder [Fig. 9·7].

Seg AA' in any of its positions is called a generator of the cylinder and the cylinder is denoted by ($ABC - A'B'C'$).

Remark. Just as a closed plane curve is the limiting form of an inscribed polygon of n sides when $n \to \infty$ such that the length of each of its sides tends to zero; similarly a cylinder can be considered as the limiting form of a prism of polygonal bases of n sides when $n \to \infty$ such that the length of each side of the n-gons tends to zero [Fig. 9·7(iv)].

(10) Some Definitions.

(i) **Altitude or Height.** The perpendicular distance between the two bases of a prism (cylinder) is called the *altitude* or *height* of the prism (cylinder).

(ii) **Right and Oblique Prisms.** A prism (cylinder) whose lateral edges (generators) are perpendicular to its bases is called a *right* prism (cylinder) [Fig. 9·7 (iv)]. A prism (cylinder) which is not right is called oblique. [Fig. 9·7 (ii)]

(iii) **Axis of a cylinder.** The line segment joining the centres of the bases of a cylinder is called the *axis* of the cylinder.

(iv) **Right Section.** A section of a prism (cylinder) by a plane perpendicular to its lateral edges (generators) is called its *right section.* A section which is not a right section is called an *oblique section.*

(v) **A regular prism.** A regular prism is a prism all of whose faces are congruent polygons. The only regular prism is a *cube.* We note that a cube is also a right prism. A cube can, therefore, be also called a *right regular prism.*

Note. However, a prism whose bases are regular polygons is often called a *regular prism.*

(vi) **A circular cylinder.** A circular cylinder is a cylinder whose bases are congruent circles.

(vii) **A right circular cylinder.** A right circular cylinder is a circular cylinder whose generators are perpendicular to its circular bases.

(viii) **A truncated prism (cylinder).** A portion of a prism (cylinder) included between a base and its section by a plane not parallel to the base is called a *truncated prism* (cylinder).

(It will be seen that the altitude of a right prism is equal in length to its lateral edge, and that of a right cylinder equals, in length, its generator.)

(11) Lateral faces of a right Prism.

Theorem. The Lateral faces of a right prism are rectangles. We have seen earlier that the lateral faces of a prism are all parallelograms.

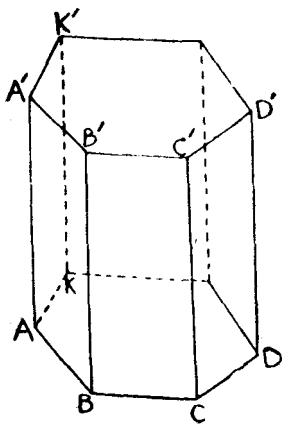


Fig. 9.8

If the prism $[ABC...K - A'B'C'...K']$ be a right prism then seg $AA' \perp$ plane of the base polygon $ABC...K$;

$\therefore$ seg $AA' \perp$ seg AB;

$\therefore$ parallelogram $ABB'A'$ is a rectangle.

Similarly, all the lateral faces of a right prism are rectangles.

It will be seen that a plane, perpendicular to a lateral edge (generator) of a prism (cylinder) will be perpendicular to all of them since all the lateral edges (generators) are parallel to one another.

(12) Section of a right prism by a plane parallel to its base.

Theorem. A section of a right prism by a plane parallel to its base is congruent to its base and is similarly placed.

Let a plane $\propto$ parallel to the base polygon $ABC \ldots K$ of the right prism $[ABC \ldots K - A'B'C' \ldots K']$ and, therefore, perpendicular to the lateral edges seg AA', seg BB', ..., seg KK' cut these segments (or segments produced) in the points A_1, B_1,..... ..., K_1, respectively (see Fig. 9·9).

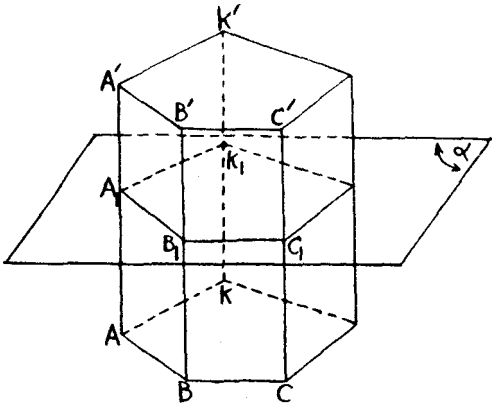


Fig. 9.9

Then the polygon $A_1B_1C_1 \ldots K_1$ is a right section of the given prism. Now AB and A_1B_1 being the intersections of two parallel planes (the plane of the polygon $ABC \ldots K$ and the plane $\propto$) by the plane of the parallelogram $ABB'A'$, we have seg $A_1B_1 \parallel$ seg AB; also seg $AA_1 \parallel$ seg BB_1.

$\therefore$ ABB_1A_1 is a parallelogram. $\therefore$ $\overline{A_1B_1} = \overline{AB}$;

Similarly, $\overline{B_1C_1} = \overline{BC}$, $\overline{C_1D_1} = \overline{CD}$ etc.

$\therefore$ polygon $A_1B_1C_1 \ldots K_1 \underline{\underline{\Omega}}$ polygon $ABC \ldots K$.

Thus every plane section of a right prism by a plane parallel to its bases is congruent to its base and is similarly placed.

You can now prove the same result for an oblique prism, viz.

Every plane section of an oblique prism by a plane parallel to its base is congruent to its base and is similarly placed.

Remarks. If $A_1B_1C_1 \ldots K_1$ and $A_2B_2C_2 \ldots K_2$ are sections of a prism by parallel planes cutting AA' in A_1, A_2; BB' in B_1, B_2;...respectively then polygon $A_1B_1C_1 \ldots K_1 \underline{\underline{\Omega}}$ polygon $A_2B_2C_2 \ldots K_2$.

The corresponding results for a cylinder (considered as the limiting form of a prism) can be stated as follows :

(i) A section of a cylinder by a plane parallel to its base is congruent to the base, and is similarly placed.

(ii) Sections of a cylinder by parallel planes are themselves congruent and are similarly placed.

(13) Surface areas of Prisms and Cylinders.

(i) Prism : Lateral surface of a prism is equal to the product of its lateral edge and the perimeter of its right section.

Let $[ABC..K-A'B'C'..K']$ be a prism and let $A_1B_1C_1...K_1$ be a right section of the prism. (see Fig. 9·9)

The lateral surface area = sum of the areas of the parallelogram faces

$$= AA' \times A_1B_1 + BB' \times B_1C_1 + ...$$
$$= l \times [A_1B_1 + B_1C_1 + ...]$$

where l is the length of the lateral edges seg AA', seg BB', ...

= l × (perimeter of a right section)

= (lateral edge) × (perimeter of a right section).

In the case of a right prism, a right section is congruent to a base of the prism. Hence the lateral surface area of a right prism

= (lateral edge) × (perimeter of a base).

The total surface area of the prism = the sum of the areas of its bases + its lateral surface area.

(ii) Cylinder : The curved surface of a right circular cylinder of radius r and height h is $2\pi rh$. (See Fig. 9·10).

Inscribe in the given cylinder a regular right polygonal prism $[ABC..K-A'B'C'..K']$ whose bases are congruent regular n-gons.

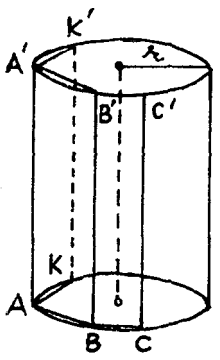


Fig. 9·10

Then the lateral surface of this prism

= sum of the areas of the rectangles $ABB'A'$, $BCC'B'$ etc.

= AA' × (perimeter of the n-gon $ABC...K$).

Now as n is gradually increased, this perimeter approaches more and more the circumference of the circle $ABC..K$ until at last when n tends to infinity the perimeter tends to the circumference of the circle. Hence the lateral surface of the cylinder, being the limiting form of the lateral surface of the inscribed prism $= h \times 2\pi r = 2\pi rh$.

Alternative Method : The lateral curved surface of a right circular cylinder can also be obtained in the following way. (See Fig. 9·11)

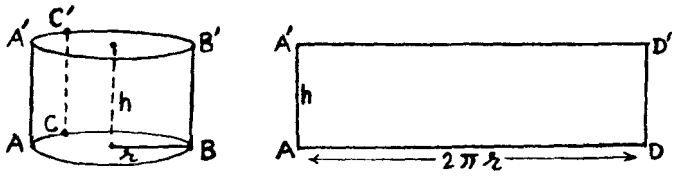


Fig. 9.11

Let r be the base radius of the cylinder and let h be its height. Cut the cylinder along a generator, say, seg AA' and develop it on a plane. Its development is the rectangle $AA'D'D$. Now seg AD is the perimeter of the bottom base of the cylinder and its length is $2\pi r$.

$\therefore$ Area of the lateral surface of the cylinder
= Area of the rectangle $AA'D'D$
$= AD \cdot AA' = 2\pi rh.$

Cor. 1. The total surface of the cylinder
$= 2\pi rh + 2\pi r^2 = 2\pi r\ (r + h).$

Cor. 2. The curved surface of a truncated right circular cylinder (Fig. 9·12) $= \pi r\ (h_1 + h_2) = 2\pi rh$ where h_1, h_2 are the lengths of the greatest and least generators, h is the distance between the centres of these bases and r is the base radius.

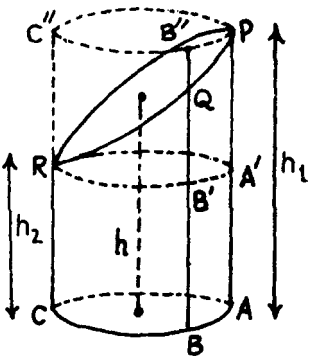


Fig. 9.12

[Let $(ABC - PQR)$ be the truncated cylinder and seg AP, seg CR, the greatest and least generators. Produce the cylinder till it meets the plane through P parallel to the base ABC. Through R draw the right section $RB'A' \parallel$ to pl. CBA, and observe that the plane PQR divides the cylinder ($A'B'R - PB''C''$) into two equal parts.]

Cor 3. The lateral surface of a circular cylinder is equal to the product of a generator and the perimeter of a right section.

(14) Surface areas of two similar right circular cylinders.

Theorem. *The surface areas of two similar right circular cylinders are to each other as the squares of their corresponding linear dimensions.*

Let $(ABC - A'B'C')$ and $(A_1B_1C_1 - A_1'B_1'C_1')$ be two similar circular cylinders of base radii r and r' and heights h and h' respectively, so that $\frac{r}{r'} = \frac{h}{h'} = k$, say.

$$\therefore \quad \frac{r+h}{r'+h'} = \frac{kr'+kh'}{r'+h'} = k = \frac{r}{r'} = \frac{h}{h'}.$$

Now, $\dfrac{\text{surface area of } (ABC - A'B'C')}{\text{surface area of } (A_1B_1C_1 - A_1'B_1'C_1')}$

$$= \frac{2\pi rh + 2\pi r^2}{2\pi r'h' + 2\pi r'^2} = \left(\frac{r}{r'}\right) \cdot \left(\frac{r+h}{r'+h'}\right) = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

Hence the theorem.

Illustrative Examples

Ex. 1. $[ABC - A'B'C']$ is a right triangular prism in which $\triangle ABC$ is equilateral If $AB = 6$ cms. and $AA' = 5$ cms., find the total surface area of the prism.

The area of the two bases of the prism $= 2$ area $(\triangle ABC) = 2\left(\frac{1}{2}ah\right)$ where $a = 6$ and $h = 6 \sin 60°$

$= 2 \times \frac{1}{2} \times 6 \times 3\sqrt{3} = 18\sqrt{3}$ sq. cms.

Lateral surface area $= 3 \times 6 \times 5 = 90$ sq. cms.

$\therefore$ Total surface area $= 18(5 + \sqrt{3})$ sq. cms.

Ex. 2. $ABCDEF$ is a base of a right (regular) hexagonal prism of base edge 10 cms. and height 6 cms. Find its total surface area.

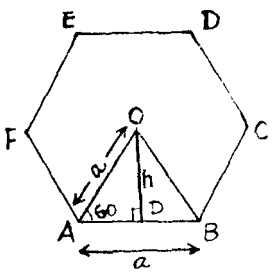


Fig. 9.13

Let O be the centre of the regular hexagonal base.

Area $ABCDEF = 6$ (Area $\triangle OAB$)

$= 6\left(\frac{1}{2}ah\right)$ where

$a = 10$ and $h = 10 \sin 60°$.

$\therefore$ The area of the regular hexagonal bases

$= 2 \times 6 \times \frac{1}{2} \times 10 \times 5\sqrt{3} = 300\sqrt{3}$ sq. cms.

Lateral surface area $= 6 \times 10 \times 6 = 360$ sq. cms.

$\therefore$ Total surface area $= 60(6 + 5\sqrt{3})$ sq. cms.

Ex. 3. $[OAC_1B - CB_1VA_1]$ is a rectangular solid in which $OA = 5$ cms., $OB = 6$ cms. and $OC = 8$ cms. Find the total surface area of the prism $[OAC_1 - CB_1V]$.

The sum of the areas of the two bases

$= 2 \times \frac{1}{2} OA \times AC_1 = 30$ sq. cms.

Area (rect. OAB_1C) $= 5 \times 8 = 40$ sq. cms.

Area (rect. AC_1VB_1) $= 6 \times 8 = 48$ sq. cms.

Area (rect. OC_1VC) $= OC_1 \times OC$

$= \sqrt{5^2 + 6^2} \times 8 = 8\sqrt{61}$ sq. cms.

$\therefore$ Total surface area

$= 118 + 8\sqrt{61} = 2(59 + 4\sqrt{61})$ sq. cms.

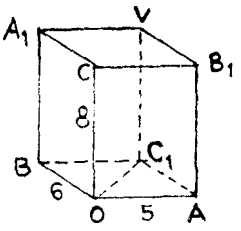


Fig. 9·14

Ex. 4. A copper wire 4 mm. in diameter is evenly wound round a cylinder whose length is 24 cms. and diameter 20 cms. so as to cover the whole surface. Find the length of the wire.

A round of wire will cover the surface 4 mm. in height. Hence the number of rounds required to cover the whole surface

$$= \frac{10}{4} \times 24 = 60.$$

$\therefore$ length of the wire $= 60 \times 2\pi \times 10$

$= 1200\pi$ cms.

Ex. 5. Find the whole surface of a right hollow cylinder whose external diameter is 14 cms., thickness 2 cms. and height 20 cms.

Whole surface $= 2\pi(7^2 - 5^2) + 2\pi \times (7 + 5) \times 20 = 528\pi$ sq. cms.

(15) Volumes of prisms and cylinders.

(i) Theorem. *Volume of a right prism is the product of its base area and height.*

Let $[ABC \ldots K - A'B'C' \ldots K']$ be a right prism whose base $ABC \ldots K$ is a polygon of area Δ square units and in which $AA' = h$ units.

Now on each of the Δ square units of the base area we can arrange a pile of h unit cubes, one on top of another. Hence the number of unit cubes contained in the prism is $\Delta \times h =$ (area of the base) $\times$ (length of the lateral edge).

$\therefore$ Volume of a right prism = (area of base) $\times$ height, since the height and a lateral edge are congruent.

(ii) Theorem. *The volume of a right circular cylinder of base radius r and height h is $\pi r^2 h$.* (Fig. 9·10)

Inscribe in the given cylinder a regular right polygonal prism $[ABC..K - A'B'C'..K']$ whose bases are congruent regular n-gons. Then $AA' = h$.

Now when n becomes indefinitely large, the polygon $(ABC..K)$ tends to equality with the circle $(ABC..K)$ and the volume of the prism tends to equality with the volume of the cylinder.

The volume of the cylinder, being the limiting value of the volume of the inscribed prism, we have the volume of the cylinder

$$= \text{Lt.}_{n\to\infty} (\text{area of the polygon } ABC..K) \times h$$
$$= (\text{area of the circle } ABC..K) \times h$$
$$= \pi r^2 h.$$

(iii) Theorem : The Volume of an oblique prism is equal to its right section multiplied by its lateral edge.

We will prove this result for a pentagonal oblique prism but the method is the same for any oblique prism. (See Fig. 9.15)

Let $[ABCDE - A'B'C'D'E']$ be an oblique prism in which $AA' = BB' = \ldots = EE' = l$, say.

Let $PQRST$ and $P'Q'R'S'T'$ be its two right sections at a distance l apart so that

$PP' = QQ' = RR' = SS' = TT' = l$.

Then $AP = A'P'$; $BQ = B'Q'$ etc.

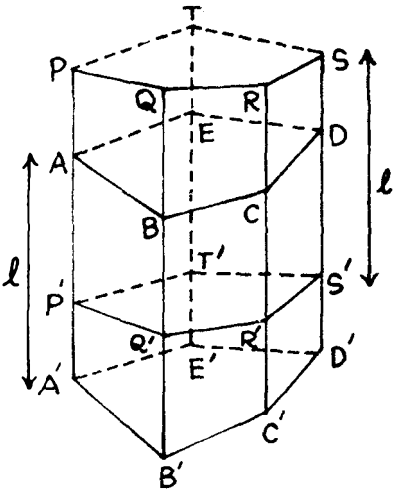


Fig. 9.15

Now in the quadrilaterals $APQB$ and $A'P'Q'B'$, we have

$\overline{AP} = \overline{A'P'}$, $\overline{PQ} = \overline{P'Q'}$, $\overline{QB} = \overline{Q'B'}$, $\overline{BA} = \overline{B'A'}$;

$\therefore$ quad. $ABQP \underline{\underline{\Omega}}$ quad. $A'B'Q'P'$.

Similarly, quad. $BCRQ \underline{\underline{\Omega}}$ quad. $B'C'R'Q'$ etc.

Also pent. $ABCDE \underline{\underline{\Omega}}$ pent. $A'B'C'D'E'$

and pent. $PQRST \underline{\underline{\Omega}}$ pent. $P'Q'R'S'T'$.

Hence in the two truncated right prisms $[ABCDE - PQRST]$, $[A'B'C'D'E' - P'Q'R'S'T']$, the corresponding bases are congruent and so also the corresponding lateral faces are congruent. Further, the corresponding lateral faces lie in the same plane (e. g. the faces $ABQP$, $A'B'Q'P'$ lie in the same plane $ABQ'P'$); also pl. $ABCDE \parallel$ pl. $A'B'C'D'E'$ and pl. $PQRST \parallel$ pl. $P'Q'R'S'T'$.

Hence the coresponding dihedral angles are of the same measure.

Hence the two truncated right prisms can be made to coincide with each other and hence are equal in volume.

Adding to each of the equal truncated right prism the part $ABCDE - P'Q'R'S'T'$, we have

vol. $[ABCDE - A'B'C'D'E'] =$ vol. $[PQRST - P'Q'R'S'T']$.

But $[PQRST - P'Q'R'S'T']$ is a right prism.

$\therefore$ vol. $[PQRST - P'Q'R'S'T'] =$ area (pent. $PQRST$) $\times$ (PP')

$\therefore$ vol. $[ABCDE - A'B'C'D'E'] =$ area (pent. $PQRST$) $\times$ l

= area (right section) $\times$ (lateral edge).

Cor. The volume of an oblique circular cylinder

= (right section) $\times$ (generator).

For its volume is the limit to which the volume of the inscribed regular oblique prism tends when the number of its lateral faces te ds to infinity in such a way that the area of the largest lateral face tends to zero. Clearly the result holds good for any cylinder.

Before proceding to obtain another formula for the volume of an oblique prism we prove the following Cavalieri's principle.

(16) Cavalieri's Principle and its use in obtaining the volume of an oblique prism

Cavalieri's Principle : Two solids, which lie between two parallel planes and are such that any two of their sections made by every plane parallel to the two given planes are equal in area, must be equal in volume.

To prove this we divide the solids into a very large number of equally thin slices by planes parallel to the given planes. Considering two corresponding thin slices, we observe that they have bases of equal area and they have the same thickness. Further, being thin, they can be considered as thin cylindrical plates. Hence they are equal in volume.

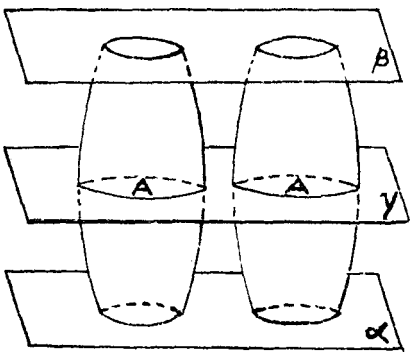


Fig. 9.16

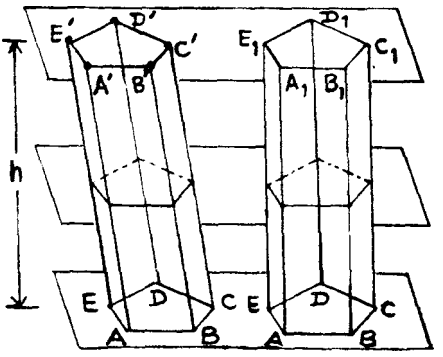
Adding together the volumes of all such slices of the two solids, it follows that the two solids are of the same volume. [Fig. 9·16]

Hence the result. We now use this Cavalieri's principle to prove the following Theorem.

Theorem : The volume of an oblique prism is equal to the product of its base and height

Let [$ABCDE-A'B'C'D'E'$] be an oblique prism [Fig. 9·17 (i)] of height h and let [$ABCDE-A_1B_1C_1D_1E_1$] be a right prism [Fig. 9·17(ii)] whose top base $A_1B_1C_1D_1E_1$ lies in the plane $A'B'C'D'E'$, their bottom bases which are congruent polygons being in the same plane. The two prisms thus lie between parallel planes and have the same height h.

Let Δ be the area of the base polygon $ABCDE$ of the two prisms. Every section of the two prisms by any plane parallel to the base will be of the same area Δ.



(i) Fig. 9·17 (ii)

$\therefore$ by Cavalieri's principle, the volumes of the two prisms are equal.

$\therefore$ volume of the oblique prism [$ABCDE-A'B'C'D'E'$]

= volume of the right prism [$ABCDE-A_1B_1C_1D_1E_1$]

= [area ($ABCDE$)] $\times$ h = $\Delta \times h$.

Thus the volume of an oblique prism is equal to the product of its base area and height.

Cor : The volume of an oblique cylinder of base radius r and height h = (Area of base) $\times$ height = $\pi r^2 h$ since the cylinder is the limiting form of the corresponding inscribed regular oblique prism when the number of its lateral faces tends to infinity in such a way that the area of every lateral face tends to zero.

Remark. If θ be the measure of the angle between the altitude h and a lateral edge (generator) l of a prism (cylinder) then it is also the angle between the planes of the base and the right section. Hence equating the two expressions for the volume of the prism (cylinder) we get

(area of right section) × (lateral edge or generator)
= (area of base) × altitude,

i. e. (area of right section) = (area of base) × $\frac{h}{l}$

= (area of base) × $\cos\theta$, since $h = l\cos\theta$.

(17) Comparison of volumes of two prisms. By using the result that the volume of a prism = (base area) × altitude, we can now prove the following theorems.

(i) Theorem : The volumes of two prisms are to each other as the product of their base-areas and altitudes.

Let $[ABC \ldots K\text{-}A'B'C' \ldots K']$, $[A_1B_1C_1 \ldots K_1\text{-}A'_1B'_1C'_1 \ldots K'_1]$ be two prisms of base areas $\triangle$ and $\triangle_1$ and altitudes h and h_1 respectively and let their volumes be V and V_1 respectively.

We have $V = \triangle h$ and $V_1 = \triangle_1 h_1$ and $\therefore \quad \frac{V}{V_1} = \frac{\triangle h}{\triangle_1 h_1}$,

and hence the result follows.

(ii) Theorem : The volumes of two similar circular cylinders are to each other as the cubes of their corresponding linear dimensions.

Let ($ABC\text{-}A_1B_1C_1$) and ($A'B'C'\text{-}A'_1B'_1C'_1$) be two similar circular cylinders of base radii r, r' and altitudes h, h' respectively,

so that $\frac{r}{r'} = \frac{h}{h'}$.

Now $\frac{\text{vol. } (ABC\text{-}A_1B_1C_1)}{\text{vol. } (A'B'C'\text{-}A'_1B'_1C'_1)} = \frac{\pi r^2 h}{\pi r'^2 h'} = \left(\frac{r}{r'}\right)^2 \left(\frac{h}{h'}\right)$

$= \left(\frac{r}{r'}\right)^2 \left(\frac{r}{r'}\right) = \frac{r^3}{r'^3} = \frac{h^3}{h'^3}$,

and so the result follows.

(iii) Theorem : Prisms having equal altitudes and having bases of equal areas have equal volumes.

In the notation of the theorem given in (i) above, if $\triangle = \triangle_1$ and $h = h_1$ then evidently $V_1 = \triangle_1 h_1 = \triangle h = V$,

and so the result follows.

Cor. 1. The volumes of prisms of equal altitudes are proportional to their base-areas.

Cor. 2. The volumes of prisms of equal base-areas are proportional to their altitudes.

Illustrative Examples.

Ex. 1. [*ABC--DEF*] is a triangular prism. Each lateral edge is 12 cm. and *AD* is inclined to the base *DEF* at 60°. If the base area is 40 sq. cms., find *h* the height, the area Δ of a right section and the volume of the prism (see Fig. 9.18).

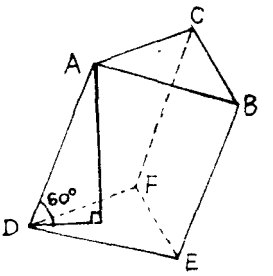


Fig. 9·18

We have $h = AD \cos 30° = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$ cms.

Also area Δ of a right section

$= (\text{area of base}) \times \cos 30° = 40 \cdot \frac{\sqrt{3}}{2}$

$= 20\sqrt{3}$ sq. cms.

$\therefore$ volume of the prism

$= 20\sqrt{3} \times 12 = 240\sqrt{3}$ cub. cms.

Ex. 2. The base of a prism of height 12 cms. is a regular hexagon *ABCDEF* of side 28 cms. Find its volume. If the lateral edges are each 16 cms. in length, find the area of a right section of the prism.

Let *O* be the centre of the regular hexagonal base. (See Fig. 9.13)

Area of *ABCDEF* = 6 (Area ΔOAB) = 6 ($\frac{1}{2} ah$) where $a = 28$ and $h = 28 \sin 60°$.

$\therefore$ area of the hexagonal base

$= 6 \times \frac{1}{2} \times 28 \times 28 \times \frac{\sqrt{3}}{2} = 1176\sqrt{3}$ sq. cms.

$\therefore$ volume of the prism

$= 1176\sqrt{3} \times 12 = 14112\sqrt{3}$ cub. cms.

If Δ be the area of a right section,

we have $\Delta \times 16 = 14112\sqrt{3}$.

$\therefore \quad \Delta = 882\sqrt{3}$ sq. cms.

Ex. 3. A big roll of paper is in the form of a right circular cylinder of radius 30 cms. The axis of the roll about which the paper is wound is also a coaxal cylinder of radius 5 cms. The length of the sheet is 220 meters. Find the thickness of the sheet.

Let *h* cms. be the height of the cylinder and *t* cms. be the thickness of the sheet. Then

$220{,}00 \times h \times t = (22/7) \times h \times (30^2 - 5^2)$.

$\therefore \quad t = \frac{22}{7} \times \frac{25 \times 35}{22000} = \frac{1}{8}$ cm.

Ex. 4. A household letter box fixed on a wall, is in the form of a truncated prism having a rectangular base of sides 5 cms. and 7 cms. The vertical edges of the box in contact with the walls are each of length 12 cms. and the remaining vertical edges are each of length 8 cms. Calculate the volume of the box. (See Fig. 9·19).

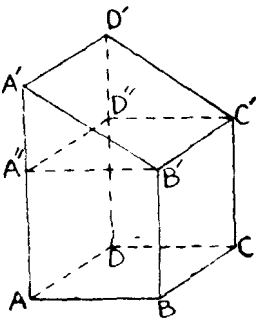


Fig. 9·19

The volume of the box

= vol. of a cuboid of edges 5, 7 and 8 cms. + volume of the triangular prism at the top $= (5 \times 7 \times 8) + (1/2) \times 5 \times 4 \times 7$
$= 280 + 70 = 350$ sq. cms.

Ex. 5. A cube is of edge 6 cms. Find the volume of the right circular cylinders inscribed in the cube and circumscribing the cube.

The radii of the circumscribing and inscribed cylinders are $3\sqrt{2}$ and 3 cms. respectively. Hence their volumes are $\pi (3\sqrt{2})^2 \times 6$ and $\pi (3)^2 \times 6$, i. e. 108π and 54π cub. cms. respectively.

EXERCISE 9 (b)

(1) The base of a right triangular prism is a triangle of sides 3, 4 and 5. If the height of the prism is 10, find its volume.

(2) The base of a right prism is a right-angled isosceles triangle. If the length of the hypotenuse of the triangle is 8 cms. and the height of the prism is 12 cms., find its volume.

(3) Find the linear dimensions of a cuboid whose volume is 210 cub. cms., surface area 214 sq. cms. and area of a base 42 sq. cms.

(4) From the four corners of a rectangular sheet of paper 80 cms. × 100 cms. four equal squares of 20 cms. side are removed and the remainder is turned into an open box. Find its volume.

(5) The base of a right prism is a trapezium whose parallel sides are 11 cms. and 7 cms. respectively, the distance between them being 6 cms. If the height of the prism be 2 cms., find its volume.

(6) A canvas tent has a floor 4 m. square. Its top consists of a horizontal ridge $3\frac{1}{2}$ m. high. The back and front are vertical and the side walls are vertical upto a height of 2 m. from the ground. Find the area of the canvas required and also the cubical content of the tent.

(7) The base of an oblique prism is a regular hexagon of side 8 cms. and its lateral edges are each of length 10 cms. If the measure of the angle between its altitude and a lateral edge be 30°, find the area of its right section and also the volume.

(8) ABC is an equilateral triangle of side 16 cms. and M is the mid-point of seg BC. If the whole triangle is rotated completely about the side AB, find the curved surface generated by seg CM.

(9) A cube of iron whose edge is 8 cms. is wholly immersed in a long cylindrical tube, filled with water, whose diameter is 20 cms. By how much will it raise the water ?

(10) A right circular cylinder of wood, 100 cms. long and 20 cms. in diameter, is sawn along two planes parallel to each other and parallel to the axis of the cylinder, each plane being at a distance from the axis equal to one half the radius of the cylinder. Find the volumes of the logs so formed.

(11) Find the whole surface of a hollow cylinder, open at both ends, if its length is 12 cms., the external diameter 8 cms. and thickness 2 cms. [Take $\pi = 22/7$.]

(12) A rectangular sheet of paper 22 cms. long and 12 cms. wide can be curved so as to form the lateral surface of two different right circular cylinders. Find the difference between the volumes of the two different cylinders which can be so formed.

(13) What quantity of a sheet of iron is required to make a cylindrical tunnel 2 m. in diameter and 40 m. long ?

(14) A rectangle $ABCD$ with $AB = 6$ cms. and $BC = 4$ cms. is revolved about seg BC. Find the area of a plane section through the axis of revolution. Find also the area of the axial section if the rectangle $ABCD$ rotates about AB.

(15) The base radius and height of a right circular cylinder are 3 cms. and 10 cms. respectively. A right prism with equilateral triangles as bases is incribed in the cylinder. Find the sum of the areas of all the faces of the prism.

(16) A cylinder of base radius 6 cms. and height 10 cms. is divided into two parts by a plane parallel to the axis and 3 cms. away from the axis. If V_1 and V_2 are the volumes of the two parts, find the ratio $V_1 : V_2$.

(17) A cylindrical drainage pipe having external and internal diameters 6 and 5 cms. respectively is to be made of clay. Find the weight of such a pipe of length 2 meters, given that the density of clay is 3 grams/c. c.

(18) Fill in the blanks in the following :

(i) A pentagonal prism has..7...faces and..15..edges.

(ii) The base radius of a right circular cylinder of height h is r. Hence the area of its section by a plane through its axis is...2. r^2

(iii) If the bases of a prism are parallelograms, then the prism is called a...parallelepiped

(iv) A prism whose lateral edges are perpendicular to its bases is called a ...right prism

(v) If two triangular prisms have the same base areas, then their volumes are proportional to their...base

(19) State whether the following statements are true or false.

(i) All edges of a right prism are congruent.

(ii) All lateral edges of a prism are parallel and equal to one another.

(iii) Every plane section of a prism is a polygon congruent with the polygonal bases of the prism.

(iv) A triangular prism has no diagonal.

(v) All lateral faces of a prism are congruent parallelograms.

(20) Choose the correct answer from the given alternatives in each of the following.

(i) The total surface area of a closed right cylinder of base radius a and height h is

(a) $\pi a^2 h$, (b) $2\pi a\,(a + h)$, (c) $\pi a^2 + \pi ah$, (d) $\pi a^2 + 2\pi ah$.

(ii) C_1 and C_2 are two similar right circular cylinders. If the radius of C_2 is 9 cms, and the ratio of the lateral surfaces of C_1 and C_2 is 25/36, then the radius of C_1 is

(a) 6 cms., (b) 7.5 cms., (c) 4.5 cms.

(iii) The volume of a right circular cylinder is 648π c. cms. and its height is 8 cms; therefore its total surface area is

(a) 144π, (b) 225π, (c) 306π, (d) none of these.

ANSWERS

(1) 60 c. units. (2) 192 c, units. (3) 5, 6, 7 cms. (4) 48000 c. cms.

(5) 108 cms.

(6) 58 sq. ms, 44 c. ms. (7) 144 sq. cms., 144 c.cms. (8) $96\pi\sqrt{3}$ sq. cms.

(9) $(128/25\pi)$ cms. (10) $10{,}000\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$ c. cms.;

$10{,}000\left(\frac{\pi}{3} - \frac{3}{4}\right)$ c. cms.; $10{,}000\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$ c. cms.

(11) 528 sq. cms. (12) $(660/\pi)$ c. cms. (13) 80π sq. ms.

(14) 48 sq. cms; 48 sq. cms. (15) $207\sqrt{3}/2$ sq. cms.

(16) $(4\pi - 3\sqrt{3}) : (8\pi + 3\sqrt{3})$. (17) 1650π gms.

(18) (i) 7, 15; (ii) $2rh$; (iii) parallelopiped, (iv) right prism, (v) heights.

(19) (i) False, (ii) True, (iii) False, (iv) True, (v) False.

(20) (i) b; (ii) b; (iii) c.

□ □ □

CHAPTER 10

Pyramids and Cones

(1) Pyramid : A Pyramid (as has been defined in article 2 of Chapter 9) is a polyhedron bounded by $(n+1)$ faces one of which is a polygon of n sides, the remaining n faces being all triangles having a common vertex which does not lie in the plane of the base polygon of n sides. We now give the following alternative definition.

Alternative Definition If one end A of a line segment (not necessarily of a constant length), always passing through a fixed point V, describes the perimeter of a polygon $ABC...K$, then the line segment VA generates the triangular faces of a solid, called a **pyramid.**

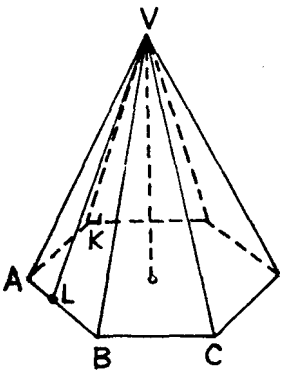


Fig. 10·1

A pyramid will be denoted by

$$[V\text{-}ABC...K].$$

In a pyramid $[V\text{-}ABC...K]$, the following terms should be remembered.

(i) The polygon $ABC...K$ is called the **base** and the fixed point V is called the **vertex** of the pyramid.

(ii) The moving line segment VA in any of its positions is called a **generator** of the pyramid.

(iii) The various triangular faces, viz VAB, VBC,...etc. having the common vertex V, are called the **lateral faces** of the pyramid.

(iv) The intersections of the consecutive lateral faces viz. VA, VB...etc. are called the **lateral edges.**

(v) The length of the perpendicular from the vertex V to the base is called the **altitude** or **height.**

(vi) A pyramid whose vertex lies on the perpendicular to its base at its centre is called a **right pyramid.**

Remark : The lateral edges of a right pyramid are equal and hence the lateral faces of a right pyramid are isosceles triangles.

(vii) A pyramid whose base is a regular polygon and in which the perpendicular from the vertex to its base passes through the centre of the base is called a **regular pyramid.**

Remarks : (i) A regular pyramid is a right pyramid but every right pyramid is not a regular pyramid; for example, if the perpendicular from the centre of a rectangular base passes through the vertex of the pyramid, then the pyramid will be a right pyramid but not a regular one.

(ii) The lateral faces of a regular pyramid are congruent isosceles triangles.

(viii) The altitude of a lateral triangular face of a pyramid is called the **slant height of that face.**

(ix) The altitude of any of the congruent isosceles triangular lateral faces of a regular pyramid is called the **slant height of the regular pyramid.**

(2) Cone. Definition : If one end A of a line, always passing through a fixed point V, describes the perimeter of a closed plane curve ABC, then the line segment VA (not necessarily of a constant length) generates the lateral surface of a solid, called a cone.

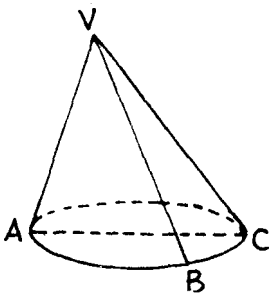


Fig. 10·2

The cone is denoted by V–ABC, the closed plane curve ABC is called the **base** of the cone and the fixed point V is called the **vertex** of the cone.

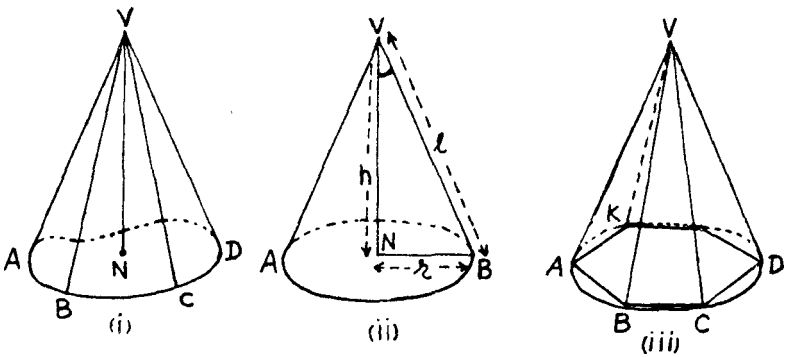


Fig. 10·3

The base of the cone need not be a circle [Fig. 10·3 (i)]. However if the base of the cone is a circle then it is called a **circular cone.** (Fig. 10·2).

A cone whose base is a circle and whose vertex lies on the perpendicular to the circular base through its centre is called a **right circular cone** [Fig 10·3 (ii)].

The line segment joining the vertex of the right circular cone to the centre of its base is called the **axis of the cone.**

The angle made by the axis with any of its generators is called the **semi-vertical angle** of the cone.

The perpendicular from the vertex to the base of the cone is called **its altitude or height.**

Remark : Just as a closed plane curve is the limiting form of an inscribed polygon of n sides when $n \to \infty$ in such a way that the length of each of its sides tends to zero, similarly a cone can be considered as the limiting form of a pyramid with a polygonal base of n sides when $n \to \infty$ such that the length of each side of the n-gon tends to zero. Fig. 10·3 (iii).

A section of a pyramid (cone) by a plane perpendicular to its altitude is called its **right section.** A section, which is not a right section, is called an **oblique section.** It may be noted that a right section of a pyramid (cone) is always parallel to its base. (Fig. 10·5 on p. 178).

A **frustum** of a pyramid (cone) is the portion of the pyramid (cone) included between the plane of its base and a plane parallel to the base.

[$ABC \quad K—A'B'C'...K'$] is a frustum of the pyramid in Fig. 10·5.

(3) Surface Area of a Pyramid.

(i) The total surface area of a pyramid = area of the base + the sum of the areas of the lateral faces, which are all triangles.

(ii) In the case of a regular pyramid, the lateral surface area takes a simple form.

In this connection we prove the following theorem :

Theorem : The lateral surface area of a regular pyramid is equal to one half the product of its slant height and the perimeter of the base.

Let [$V–ABC...K$] be a regular pyramid whose base is the regular polygon $ABC...K$ and whose slant height, i.e. the altitude of each of the congruent isosceles triangles, is l (see Fig 10·4).

Thus if seg VL be $\perp$ to AB then $VL = l$.

Now (area of $\triangle$ VAB)

$$= \frac{1}{2} VL \times AB$$

$$= \frac{1}{2} l \times AB.$$

Similarly, the area of each of the other lateral faces which are congruent isosceles triangles

$$= \frac{1}{2} l \times (\text{base}).$$

$\therefore$ the lateral surface area

= (area of $\triangle$ VAB) + (area of $\triangle$ VBC) + ...

$$= \frac{1}{2} l \times AB + \frac{1}{2} l \times BC + \ldots.$$

$$= \frac{1}{2} l \times [\, AB + BC + \ldots. \,]$$

$$= \frac{1}{2} l \times \text{perimeter of the base}$$

$$= \frac{1}{2} (\text{slant height}) \times (\text{perimeter of the base}).$$

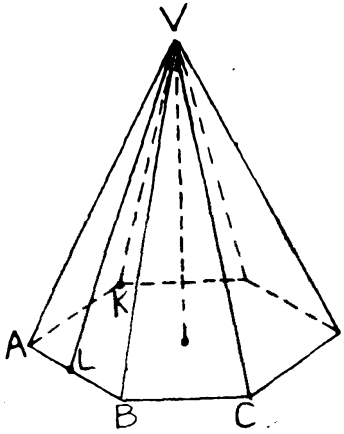


Fig. 10·4

(4) Theorem : A section of a pyramid, by a plane parallel to its base, is similar and is similarly placed to the base.

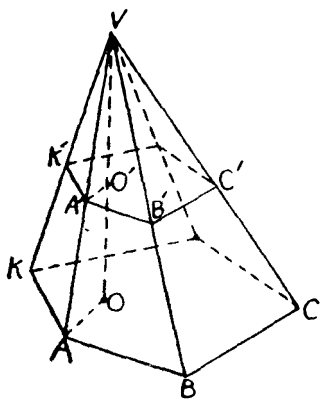


Fig. 10·5

Let a plane π parallel to the base of the pyramid [$V - ABC...K$] cut the lateral edges seg VA, seg VB, seg VC,... in the points A',B',C', ... respectively. Clearly $A'B'$ and AB being sections of the parallel planes π and pl. ABC...respectively by the cutting pl. VAB, (see Fig. 10·5),

seg $A'B' \parallel$ seg AB.

$\therefore$ from $\triangle$ VAB,

$$\frac{VA'}{A'A} = \frac{VB'}{B'B}.$$

Thus the lateral edges seg VA and seg VB are cut by the plane π proportionally.

Further, if the altitude seg VO cuts the plane π in O', we have by considering the triangle VOA in which seg $O'A' \parallel$ seg OA,

$$\frac{VO'}{O'O} = \frac{VA'}{A'A}.$$

It thus follows that all the lateral edges and the altitude are cut proportionally by the plane π parallel to the base of the pyramid.

Further as seen above

seg $A'B' \parallel$ seg AB, seg $B'C' \parallel$ seg BC, seg $C'D' \parallel$ seg CD etc.

$\therefore$ $A'B'C'...K'$ and $ABC...K$ are equiangular polygons.

Also $\frac{A'B'}{AB} = \frac{VA'}{VA} = \frac{VB'}{VB} = \frac{B'C'}{BC} = ..$ etc.

$\therefore$ $A'B'$, $B'C'$, $C'D'$, ... etc. are proportional to AB, BC, CD, ... etc. respectively.

$\therefore$ section $A'B'C'...K'$ is similar and similarly placed to the section $ABC...K$.

(5) Comparison of Areas of Parallel Sections of a Pyramid.

Theorem : The areas of parallel sections of a pyramid are to each other as the squares of their distances from the vertex. (Fig. 10·6)

Let [V-$ABC...K$] be a pyramid. Let parallel planes π_1 and π_2 cut the edges seg VA, seg VB, seg VC, ... in the points A_1, B_1, C_1, ... and A_2, B_2, C_2, ... respectively so that $A_1B_1C_1...K_1$ and $A_2B_2C_2...K_2$ are parallel sections. Let seg VO_2 be $\perp$ to the section $A_2B_2C_2...K_2$ at O_2 and let it cut the section $A_1B_1C_1...K_1$ in O_1. Then $VO_1 = h_1$ and $VO_2 = h_2$ are the distances of the two sections (by the parallel planes π_1 and π_2) respectively from the vertex V. Join O_1, A_1 and O_2, A_2.

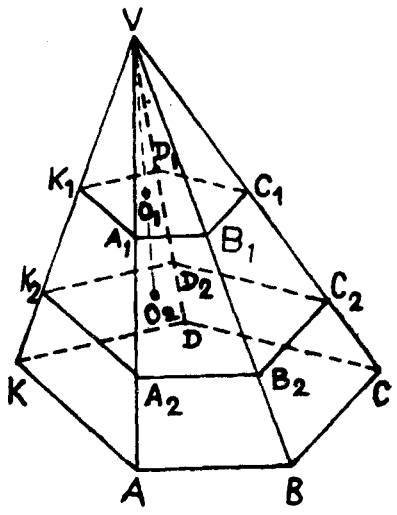


Fig.-10·6

Now by the theorem on intersections of parallel planes by a cutting plane, we have seg $A_1B_1 \parallel$ seg A_2B_2, seg $B_1C_1 \parallel$ seg B_2C_2 etc.; also seg $O_1A_1 \parallel$ seg O_2A_2.

Also the corresponding sides of the sections can be shown to be proportional.

$\therefore$ section $A_1B_1C_1...K_1$ and section $A_2B_2C_2...K_2$ are polygons, similar and similarly placed.

$$\therefore \quad \frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1D_1}{C_2D_2} = \text{, say, } \frac{l_1}{l_2}, \qquad \ldots \quad \ldots \quad \text{(i)}$$

where l_1 and l_2 are the lengths of the corresponding sides.

By joining A_1C_1, A_1D_1, A_2C_2, A_2D_2, the similar polygons $A_1B_1...K_1$ and $A_2B_2...K_2$ can be divided into pairs of similar triangles $A_1B_1C_1$, $A_2B_2C_2$; $A_1C_1D_1$, $A_2C_2D_2$; etc.

From the pair of similar triangles $A_1B_1C_1$ and $A_2B_2C_2$,

we have $$\frac{\text{area of } \triangle\ A_1B_1C_1}{\text{area of } \triangle\ A_2B_2C_2} = \frac{A_1B_1^2}{A_2B_2^2} = \frac{l_1^2}{l_2^2}.$$

$\therefore$ area of $\triangle\ A_1B_1C_1 = (l_1^2/l_2^2)$. area of $\triangle\ A_2B_2C_2$.

Similarly area of $\triangle\ A_1C_1D_1 = (l_1^2/l_2^2)$. area of $\triangle\ A_2C_2D_2$.

Adding all such results, we get

area of poly. $A_1B_1C_1...K_1 = (l_1^2/l_2^2)$ area of poly. $A_2B_2C_2...K_2$ ... (ii)

Now in $\triangle\ VA_2B_2$, $A_1B_1 \parallel A_2B_2$;

$$\therefore \frac{VA_1}{VA_2} = \frac{A_1B_1}{A_2B_2} = \frac{l_1}{l_2}. \qquad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Similarly in $\triangle\ VO_2A_2$, $O_1A_1 \parallel O_2A_2$;

$$\therefore \frac{VO_1}{VO_2} = \frac{VA_1}{VA_2} = \frac{l_1}{l_2}, \text{ on using (iii).} \qquad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

From (ii) and (iv), we get

$$\frac{\text{area of poly. } A_1B_1C_1...K_1}{\text{area of poly. } A_2B_2C_2...K_2} = \frac{l_1^2}{l_2^2} = \frac{h_1^2}{h_2^2}. \quad \text{Hence the result.}$$

Cor. 1 : If the section $A_1B_1C_1...K_1$ be by a plane parallel to the base $ABC...K$ and at a distance x from V and h be the altitude of the pyramid, we have

$$\frac{(\text{area of section } A_1B_1C_1...K_1)}{x^2} = \frac{(\text{area of base } ABC...K)}{h^2}.$$

Cor. 2. : If $A_1B_1C_1$ be a section of a tetrahedron $[V - ABC]$ by a plane parallel to the base ABC and x be the distance of the section

$A_1B_1C_1$ from V and h be the height of the tetrahedron, then triangles $A_1B_1C_1$ and ABC are evidently similar and similarly placed and

$$\frac{\text{area } (\triangle A_1B_1C_1)}{x^2} = \frac{\text{area } (\triangle ABC)}{h^2}.$$

This result follows by considering a tetrahedron as a particular case of a pyramid.

(6) A section of a right circular cone by a plane parallel to its base.

Theorem. *A section of a right circular cone by a plane parallel to its base divides its altitude and generators proportionally.*

Let $(V - A_1B_1C_1)$ be a right circular cone and let O be the centre of the base circle $A_1B_1C_1$ so that $VO = h$.

Inscribe in the base circle $A_1B_1C_1$ a regular n-gon $A_1A_2...A_n$ and let a plane π, parallel to the base, cut seg VA_1, seg VA_2 ....seg VA_n in the points A_1', A_2'... , A_n' respectively. (See Fig. 10-7)

Now by the theorem on intersections of two parallel planes by a cutting plane, seg $A_1'A_2'$ ∥ seg A_1A_2; seg $A_2'A_3'$ ∥ seg A_2A_3 etc.

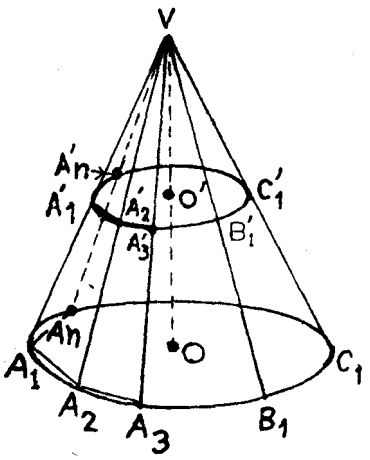


Fig. 10.7

∴ polygons $A_1A_2...A_n$ and $A'_1A'_2...A'_n$ are equiangular and can be proved to be similar and are obviously similarly placed.

Now as n tends to infinity the limiting forms of the polygons $A_1A_2...A_n$ and $A'_1A'_2...A'_n$ are the circles, say, $A_1B_1C_1$ and $A'_1B'_1C'_1$. The section of the cone by the plane π is thus a circle.

Let VO cut the plane π in O', the centre of the circle $A'_1B'_1C'_1$ and let $O'A'_1 = r'$, and $VO' = h'$.

From $\triangle VOA_1$ in which seg $O'A'_1$ ∥ seg OA_1,

we have $\frac{VA'_1}{VA_1} = \frac{VO'}{VO} = \frac{h'}{h}$.

Similarly $\frac{VA'_2}{VA_2} = \frac{h'}{h}$, etc.

Thus the plane π, parallel to the base, divides the altitude and the generators of the cone proportionally.

We now prove the following theorem on the area of a section of a right circular cone by a plane parallel to its base.

(7) Relation between the area of the base and the area of a section by a plane parallel to the base.

Theorem : *The ratio of the area of a section of a right circular cone by a plane parallel to the base to the area of the base is equal to the ratio of the square of its distance from the vertex to the square of its altitude,*

With the notation and figure of the above article, we have

$$\frac{\text{area (circle } A_1'B_1'C_1')}{\text{area (circle } A_1B_1C_1)} = \frac{\pi (O'A_1')^2}{\pi (OA_1)^2} .$$

But from the similar triangles $VO'A_1'$ and VOA_1, we have

$$\frac{O'A_1'}{OA_1} = \frac{VO'}{VO} = \frac{h'}{h} .$$

$$\therefore \quad \frac{\text{area of the section } (A_1'B_1'C_1')}{\text{area of the base } (A_1B_1C_1)} = \frac{h'^2}{h^2} .$$

Hence the theorem.

(8) Lateral surface of a right circular cone.

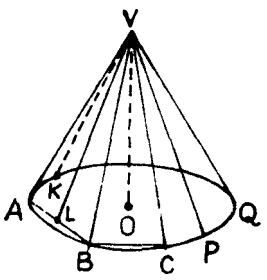


Fig. 10·8

Theorem : The lateral curved surface of a right circular cone of base-radius r and generator l is πrl. (Fig. 10·8).

Inscribe in the right circular cone $(V\text{-}APQ)$ a right regular pyramid of n lateral faces viz. $[V\text{-}ABC\ldots K]$.

Draw seg$VL \perp AB$. Then the lateral surface of the regular pyramid

= the sum of the areas of the triangular faces VAB, VBC, etc.

$= \frac{1}{2} VL \times$ (perimeter of the polygon $ABC\ldots K$).

Now when n becomes larger and larger, the polygon tends to equality with the circle $ABC\ldots K$, seg VL with the generator seg VA and the lateral surface of the pyramid with the lateral curved surface of the cone. Hence the lateral surface of the cone

$= \frac{1}{2} VA \times$ [(circumference of the circle $ABC\ldots K$), i. e. of the circle (APQ)]

$= \frac{1}{2} l \times 2\pi r = \pi rl.$

Alternative Method. The lateral curved surface of a right circular cone ($V-ABC$) of base-radius r and generator l can be alternatively determined as follows. [See Fig. 10·9 (i) and 10·9 (ii).]

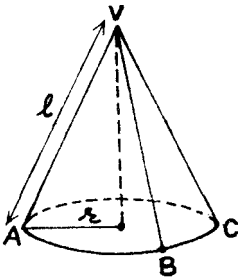


Fig. 10.9 (i)

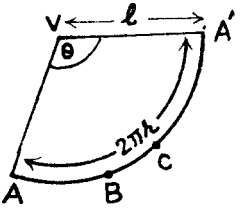


Fig. 10.9 (ii)

We cut the cone along a generator seg VA keeping the vertex V and the generator seg VA fixed and develop it on a plane. This development is a sector $VABA'$ of a circle with V as centre and, l as radius. Then arc ABA' is the perimeter of the base of the cone and is of length $2\pi r$. If θ be the measure of the angle of the sector, we get $l\,\theta = 2\pi r$.

The lateral surface area of the cone

= the area of the sector $VABA'$

$= \frac{1}{2} l^2 \times \theta$

$= \frac{1}{2} l^2 \times \frac{2\pi r}{l} = \pi r l$ as before.

Cor. 1. If α be the measure of the semi-vertical angle of the cone we have $r = l \sin \alpha$ and hence the lateral curved surface can also be expressed as $\pi l^2 \sin \alpha$ or $\pi r^2 \operatorname{cosec} \alpha$.

The total surface area of the cone

$$= \pi r^2 + \pi r l = \pi r\,(r + l).$$

Cor. 2. The lateral surface of a frustum of a right circular cone of base-radii r_1 and r_2 ($r_2 > r_1$) and generator l can be considered as the difference between the lateral surfaces of two right circular cones with a common vertex and base radii r_1 and r_2 and generators l_1, l_2 where $l = l_2 - l_1$. (See Fig. 10·10).

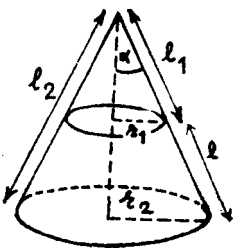


Fig. 10.10

Since $r_1 = l_1 \sin \alpha$ and $r_2 = l_2 \sin \alpha$, where α is the measure of the semi-vertical angle of the cone, we get the lateral surface of the frustum of the cone

$= \pi r_2 l_2 - \pi r_1 l_1 = \pi (l_2^2 - l_1^2) \sin \alpha$

$= \pi (l_2 \sin \alpha + l_1 \sin \alpha)(l_2 - l_1) = \pi (r_2 + r_1) l.$

$\therefore$ the total surface area of the frustum will be

$$\pi (r_1^2 + r_2^2) + \pi (r_2 + r_1) l.$$

Illustrative Examples :

Ex. 1 : A canvas tent is in the form of a right regular hexagonal prism surmounted by a pyramid with the same regular hexagonal base. A side of the common hexagonal base is $4\sqrt{3}$ m., the height of the prism 6 m. and that of the pyramid is 8 m. Show that the total amount of canvas used is $264\sqrt{3}$ sq. m.

Let $[ABCDEF - A'B'C'D'E'F']$ be the prism and $[V - A'B'C'D'E'F']$ be the pyramid. Let seg VO be $\perp$ to the base $ABCDEF$ and let it cut the plane $A'B'C'D'E'F'$ in O'.

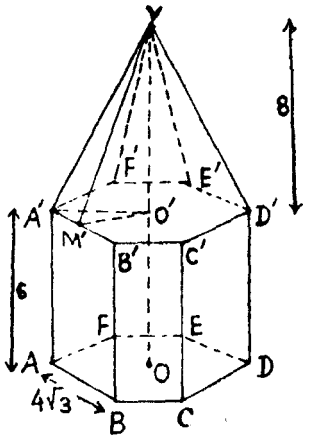


Fig. 10·11

Let seg $O'M'$ be $\perp$ to seg $A'B'$. Join VM'; then $m(\angle O'A'M') = 60°$, and $m(\angle O'M'A') = 90°$

$\therefore$ we have $O'M' = O'A' \sin 60° = 6$ m.

$\therefore$ $[VM']^2 = [VO']^2 + [O'M']^2$

$\because$ $VO' \perp O'M'$.

$= 64 + 36 = 100.$

$\therefore$ $VM' = 10$ m. = slant height.

Now lateral surface area of the pyramid

$= \frac{1}{2}$ (slant height) $\times$ (perimeter of the base)

$= \frac{1}{2} \times 10 \times 6 \times 4\sqrt{3} = 120\sqrt{3}$ sq. m.

Lateral surface area of the prism

= (height) $\times$ (perimeter of the base)

$= 6 \times 6 \times 4\sqrt{3} = 144\sqrt{3}$ sq. m.

$\therefore$ total canvas = total lateral surface area

$= 144\sqrt{3} + 120\sqrt{3} = 264\sqrt{3}$ sq. m.

Ex. 2 : A piece of paper in the form of a semi-circle of radius a just covers the lateral surface of a right circular cone. Prove that the measure of the semi-vertical angle of the cone is 30°.

We have that the radius of the semi-circle viz. a

= length of a generator of the cone = l, say.

$\therefore$ $l = a$.

Also r being the base-radius of the cone, the perimeter of the base of the cone

$= 2\pi r$ = length of the semi-circle $= \pi a$.

$\therefore$ $r = a/2$.

If α be the measure of the semi-vertical angle of the cone, we have

$$\sin \alpha = \frac{r}{l} = \frac{(a/2)}{a} = \frac{1}{2}. \quad \therefore \quad m(\alpha) \equiv 30°.$$

Ex. 3 : A right circular cone of height $\sqrt{3}$ m. is divided by planes parallel to its base into three parts of equal surface area. Find the distances of these planes from the vertex of the cone.

Let h_1 and h_2 ($h_2 > h_1$) be the distances of these planes from the vertex of the cone. Since the lateral surface of a right circular cone of height h is πrl, i. e. $\pi h^2 \tan \propto \sec \propto$ and, therefore, varies as the square of its height, we have by the data

$$\frac{h_1^2}{1} = \frac{h_2^2}{2} = \frac{(\sqrt{3})^2}{3} = 1.$$

$\therefore h_1 = 1$ m. and $h_2 = \sqrt{2}$ m.

Ex. 4 : Find the height of a right pyramid whose total surface measures $270\sqrt{3}$ sq. cms. and whose base is an equilateral triangle of side $10\sqrt{3}$ cms.

If h be the height of the pyramid and l its slant height, then

$l^2 = h^2 + (m/3)^2$, where m is the length of a median of the equilateral base triangle and is $10\sqrt{3} \times \sqrt{3}/2 = 15$ cms;

$\therefore l = \sqrt{h^2 + 25}$.

Now the whole surface = slant surface + base area

$$= 3 \times \frac{10\sqrt{3}}{2} \times \sqrt{h^2 + 25} + (\sqrt{3})\frac{100 \times 3}{4}.$$

$$\therefore 270\sqrt{3} = 15\sqrt{3} \times \sqrt{h^2 + 25} + 75\sqrt{3}.$$

$$\therefore 15 \times \sqrt{h^2 + 25} = 270 - 75 = 195.$$

$$\therefore \sqrt{h^2 + 25} = 13; \quad \therefore h = 12 \text{ cms.}$$

Ex. 5 : A frustum of a regular pyramid has square bases of 8 cms, and 5 cms. sides and its height is 2 cms. Find the total surface area.

The slant height of a lateral face

$$= \sqrt{(2)^2 + \left(\frac{8-5}{2}\right)^2} = \frac{5}{2} \text{ cms.}$$

$$\therefore \text{ total area } = 4 \times \frac{1}{2}(8 + 5) \times \frac{5}{2} + 64 + 25$$

$$= 154 \text{ sq. cms.}$$

(9) Volume of a Tetrahedron :

Theorem : The volume of a tetrahedron is $\frac{1}{3}$ of the product of its base area and length of its height.

Let $[V - ABC]$ be a $\mathcal{T}$ and let seg VO be perpendicular to the base ABC.

Let area $(\triangle ABC) = \triangle$ and let $VO = h$. We have then to prove that

vol. $[V-ABC] = (\frac{1}{3}) \triangle h$. (See Fig. 10.12)

We divide the $\mathbb{T}$ into a very large number of thin slices by planes parallel to the base ABC.

Let $[A_1B_1C_1 - A_2B_2C_2]$ be a typical slice and let seg VO cut the section $A_1B_1C_1$ in O_1 and the section $A_2B_2C_2$ in O_2. Let $VO_1 = x$ and $VO_2 = x + \delta x$ so that δx is the thickness of the thin slice. Let vol. $[A_1B_1C_1 - A_2B_2C_2] = \delta V$.

Now if the slice be very thin, it can be regarded as a prism whose height is δx and whose base is $\triangle A_1B_1C_1$.

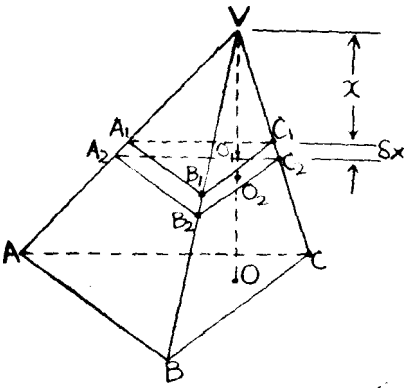


Fig. 10.12

$$\therefore \quad \delta V = \text{area}\,(\triangle A_1B_1C_1) \cdot \delta x.$$

$$= \frac{\text{area}\,(\triangle A_1B_1C_1)}{\text{area}\,(\triangle ABC)} \cdot \text{area}\,(\triangle ABC) \cdot \delta x$$

$$= \frac{x^2}{h^2} \cdot \triangle \cdot \delta x$$

$$\therefore \quad \text{vol.}\,[V-ABC] = \Sigma \delta V = \Sigma \frac{\triangle}{h^2} x^2\, \delta x = \frac{\triangle}{h^2} \int_0^h x^2\, dx.$$

$$= \frac{\triangle}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3} \triangle h = \frac{1}{3}(\text{base}) \times (\text{height}).$$

Cor : If $[V-ABC...K]$ be a pyramid of height h and base area $\triangle$, then vol. $[V-ABC...K] = \frac{1}{3} \triangle h$.

For the pyramid $[V-ABC...K]$ can be divided into a number of pyramids $[V-ABC]$, $[V-ACD]$, $[V-ADE]$ ... etc. with a common height h and base areas $\triangle_1$, $\triangle_2$, $\triangle_3$, ... whose sum is $\triangle$.

(10) Volumes of similar tetrahedra :

Theorem : The volumes of two similar tetrahedra are in the ratio of the cubes of their corresponding edges.

Let $[V-ABC]$ and $[V'-A'B'C']$ be two similar tetrahedra so that $\triangle ABC$ and $\triangle A'B'C'$ are similar. Let seg VO and seg $V'O'$ be $\perp$ to the planes containing bases of triangles ABC and $A'B'C'$ respectively.

Now $\dfrac{\text{area }(\triangle ABC)}{\text{area }(\triangle A'B'C')} = \dfrac{[AB]^2}{[A'B']^2}$

$= \dfrac{[VA]^2}{[V'A']^2}$, since triangles VAB, $V'A'B'$ are similar.

Also $\dfrac{VO}{V'O'} = \dfrac{VA}{V'A'}$, since triangles VOA and $V'O'A'$ are similar.

$\therefore \dfrac{\text{vol}\,[V-ABC]}{\text{vol}\,[V'-A'B'C']} = \dfrac{\frac{1}{3}\text{ area }(\triangle ABC) \times VO}{\frac{1}{3}\text{ area }(\triangle A'B'C') \times V'O'}$

$= \dfrac{\text{area }(\triangle ABC)}{\text{area }(\triangle A'B'C')} \times \dfrac{VA}{V'A'}$

$= \dfrac{[VA]^2}{[V'A']^2} \times \dfrac{VA}{V'A'} = \dfrac{[VA]^3}{[V'A']^3}$ etc.

$= \dfrac{e^3}{e'^3}$, where e and e' are the lengths of a pair of corresponding edges of the similar tetrahedra.

(11) Volume of a right circular cone.

Theorem : The volume V of a right circular cone of base-radius r and height h is equal to $\frac{1}{3}\pi r^2 h$.

Inscribe in the right circular cone $(V-APQ)$ a right regular pyramid $[V-ABC...K)$, of n lateral faces. (See Fig. 10·8).

Let seg VO be $\perp$ the base of the pyramid at its centre O, so that $VO = h$.

Now when n becomes indefinitely large, the polygon $ABC..K$ tends to equality with the circle $(ABC..K)$, i.e. with the circle APQ, and the volume of the pyramid with the volume of the right circular cone.

$\therefore V = \text{Lt. } \frac{1}{3}$ (area of the polygon $ABC..K$) $\times h$

$= \frac{1}{3}$ (area of the circle APQ) $\times h$

$= \frac{1}{3}\pi r^2 h$.

Alternative Method : Divide the cone into a large number of thin slices by planes parallel to the base. Let δV be the volume of a typical

thin slice bounded by planes at distances x and $x + \delta x$ from the vertex V. The area of the base of the typical thin slice at a distance x from V is $\frac{x^2}{h^2}\pi r^2$ and the slice being very thin, we have

$$\delta V = \pi r^2 \times \frac{x^2}{h^2}\delta x.$$

$$\therefore \quad \text{vol } (V-ABC) = \Sigma \delta V = \int_0^h \frac{\pi r^2 x^2}{h^2}\, dx.$$

$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{1}{3}\pi r^2 h.$$

If the cone be right circular with the measure of the semi-vertica angle equal to α, we have $r = h \tan \alpha$ and the volume of a right circular cone can also be expressed as $\frac{1}{3}\pi h^3 \tan^2 \alpha$ or $\frac{1}{3}\pi r^3 \cot \alpha$.

(12) Volumes of similar right circular cones.

Theorem : *The volumes of similar right circular cones are proportional to the cubes of their corresponding linear dimensions.*

Let $V_1, V_2, V_3, \ldots$ be the volumes of similar right circular cones of base radii $r_1, r_2, r_3, \ldots$and altitudes $h_1, h_2, h_3, \ldots$so that

$$V_1 = \tfrac{1}{3}\pi r_1^2 h_1, \quad V_2 = \tfrac{1}{3}\pi r_2^2 h_2, \quad V_3 = \tfrac{1}{3}\pi r_3^2 h_3, \ldots\ldots\text{etc.} \qquad \ldots \ (\text{i})$$

$$\therefore \quad \frac{V_1}{r_1^2 h_1} = \frac{V_2}{r_2^2 h_2} = \frac{V_3}{r_3^2 h_3} = \ldots \text{ etc.}$$

Since the cones are similar, we have

$$\frac{r_1}{h_1} = \frac{r_2}{h_2} = \frac{r_3}{h_3} = \ldots = \frac{1}{k}, \text{ say.}$$

$$\therefore \quad h_1 = kr_1, \ h_2 = kr_2, \ h_3 = kr_3 \text{ etc.} \qquad \ldots \quad \ldots \quad \ldots \ (\text{ii})$$

Eliminating $h_1, h_2, h_3, \ldots$ we get

$$\therefore \quad \frac{V_1}{r_1^3} = \frac{V_2}{r_2^3} = \frac{V_3}{r_3^3} = \ldots \text{ etc.}$$

Eliminating $r_1, r_2, r_3, \ldots \ldots \ldots$ we get

$$\frac{V_1}{h_1^3} = \frac{V_2}{h_2^3} = \frac{V_3}{h_3^3} = \ldots \text{ etc.}$$

Hence the theorem.

Illustrative Examples :

Ex. 1. Find the lateral surface and volume of a regular pyramid of height 15 cm. standing on a square base of side 16 cms.

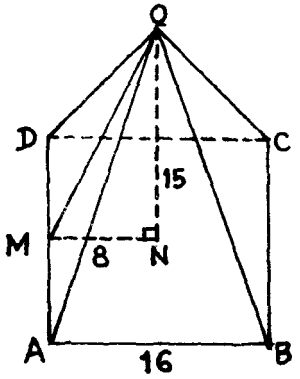


Fig. 10.13

The slant height, say OM, of the pyramid,

$= \sqrt{15^2 + 8^2} = 17$ cm.

$\therefore$ lateral surface = 4 area $\triangle OAD$

$= 4\,(1/2 \cdot AD.\ OM)$

$= 4 \times (1/2) \times 16 \times 17 = 544$ sq. cm.

Vol. $= (1/3) \times 16 \times 16 \times 15 = 1280$ cub. cm.

Ex. 2. The base of a regular pyramid is an equilateral triangle of side 4. If the number representing its whole surface is three times the number representing its volume find the number representing its height.

Let $[O-ABC]$ be a $\mathfrak{T}$ in which, each side of $\triangle ABC$ is 4.

Let D be the mid-point of BC so that $BC \perp AD$ and $OD.\ BC$ then is $\perp$ pl. (OAD). Draw seg $OX \perp AD$; then $OX = h$ is the altitude.

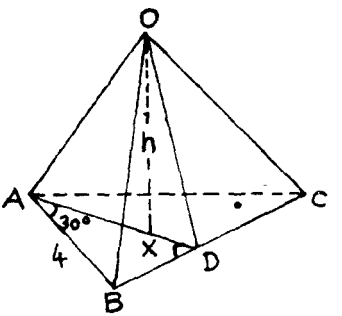


Fig. 10.14

Now $AD = 4 \cos 30° = \dfrac{4\sqrt{3}}{2} = 2\sqrt{3}$,

and $XD = \dfrac{1}{3}(AD) = \dfrac{2\sqrt{3}}{3}$.

$\therefore\ [OD]^2 = h^2 + \left[\dfrac{2\sqrt{3}}{3}\right]^2 = h^2 + \dfrac{4}{3}$.

$\therefore$ whole surface = 3 area $(\triangle OBC)$ + area $(\triangle ABC)$

$$= 3 \times \tfrac{1}{2} \times 4 \times \sqrt{h^2 + \frac{4}{3}} + \frac{16\sqrt{3}}{4}$$

and volume $= \tfrac{1}{3} \times \dfrac{16\sqrt{3}}{4} \times h$.

By data, $6 \times \sqrt{h^2 + \frac{4}{3}} + 4\sqrt{3} = 4\sqrt{3}\,h$.

$\therefore\ 36\left(h^2 + \frac{4}{3}\right) = 48\,(h-1)^2$.

$\therefore\ h^2 - 8h = 0$; i. e. $h = 0$ or 8.

Neglecting the value 0, we have $h = 8$.

Ex. 3. $ABCD$ is a square paper whose side is a; E and F are the mid-points of seg BC and seg CD. The paper is folded along AE, EF and FA. Find the volume of the pyramid thus formed.

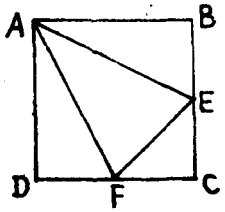


Fig. 10·15 (i)

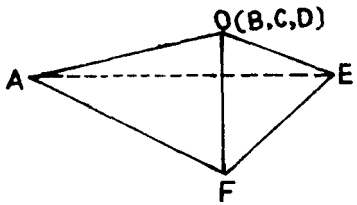


Fig. 10·15 (ii)

The base of the pyramid is $\triangle AEF$ and traingles ABE, ADF and CFE will be its lateral faces. [Fig. 10.15 (i), (ii)].

If O be the vertex, it will be the point where B, C, D meet after folding.

Clearly OA, OE, OF are perpendicular to each other.

Now $OA = AB = AD = a$,

$OE = BE = CE = a/2$,

and $OF = DF = CF = a/2$.

$\therefore$ the required volume $= \frac{1}{3}$ (area of the base) $\times$ height $= \frac{1}{3}(\triangle OAF).OE$

$$= \frac{1}{3} \times \frac{a}{2} \times \frac{a}{2} \times \frac{a}{2} = \frac{a^3}{24}.$$

Ex. 4. Find the volume of a frustum of a pyramid whose bases are of areas S_1 and S_2 the height of the frustum being h. (Fig. 10·16)

Let [$EFGH - ABCD$] be a frustum of the pyramid [$O-ABCD$] and let seg $OQ \perp$ to the base $ABCD$ cut the base $EFGH$ in P so that $PQ = h$. Let S_1 and S_2 be the areas of the two bases $ABCD$ and $EFGH$ and let $OQ = h_1$, and $OP = h_2$.

The areas S_1 and S_2 of the similar bases $ABCD$ and $EFGH$ are proportional to the squares of their distances h_1 and h_2 from the vertex O.

$$\therefore \frac{S_1}{h_1^2} = \frac{S_2}{h_2^2} = k, \text{ say. } \therefore S_1 = kh_1^2,\ S_2 = kh_2^2,\ \sqrt{S_1 S_2} = k\,h_1 h_2.$$

$$\therefore \frac{S_1}{h_1^2} = \frac{S_2}{h_2^2} = \frac{\sqrt{S_1 S_2}}{h_1 h_2} = k.$$

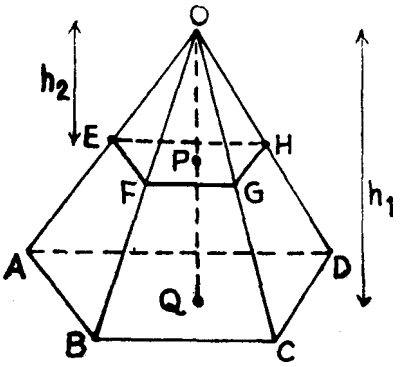


Fig 10·16

Now the volume of the frustum

$$= \frac{1}{3} S_1 h_1 - \frac{1}{3} S_2 h_2$$

$$= \frac{1}{3} k h_1^3 - \frac{1}{3} k h_2^3$$

$$= \frac{k}{3} (h_1 - h_2)(h_1^2 + h_1 h_2 + h_2^2) = \frac{h}{3} \cdot (k h_1^2 + k h_1 h_2 + k h_2^2)$$

$$= \frac{h}{3} (S_1 + \sqrt{S_1 S_2} + S_2)$$

Note : The volume of a frustum of a cone whose base-radii are r_1, r_2 and whose height is h will be

$$\frac{\pi h}{3} (r_1^2 + r_1 r_2 + r_2^2).$$

Ex. 5. Prove that the volume of a truncated triangular right prism is equal to one-third the product of its base and the sum of its lateral edges.

Let $[ABC - A'B'C']$ be the truncated triangular right prism on the base ABC so that the lateral edges are $\perp$ pl. ABC.

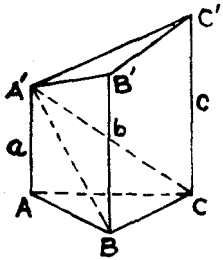


Fig. 10·17

Let $AA' = a$, $BB' = b$, $CC' = c$, $a < b < c$.

The plane $A'BC$ divides the solid into two pyramids $[A' - ABC]$ and $[A' - BCC'B']$.

Now vol. $[A' - ABC] = \frac{1}{3}$ area $(\triangle ABC) \times a$ ... (i)

and vol. $[A' - BCC'B']$

$= \frac{1}{3}$ (area of trapezium $BCC'B'$) $\times$ perpendicular distance from A' on $BCC'B'$

$= \frac{1}{3}(b + c) \times \frac{1}{2} BC \times$ (perpendicular distance from A on BC),

(since pl. $ABC \perp$ pl. $BCC'B'$)

$= \frac{1}{3}$ area ($\triangle ABC$) $\times (b + c)$. ... ... (

$\therefore$ vol. $[ABC - A'B'C'] = \frac{1}{3}$ (area $\triangle ABC$) $(a + b + c)$.

Remarks :

(i) If G and G' be the centroids of triangles ABC and $A'B'C'$ then

vol. of the truncated right triangular prism $[ABC - A'B'C']$

$=$ area ($\triangle ABC$) $\times GG'$.

(ii) The volume of any truncated triangular prism is equal to one-third the product of a right section and the sum of the lateral edges.

Ex 6. A frustum of a regular pyramid has square ends. The edge of the lower end is 10 cms. and that of the upper is 5 cms. The height is 7·5 cms. Find the length of a slant edge of the frustum and the area of the slant faces. (Fig. 10.18).

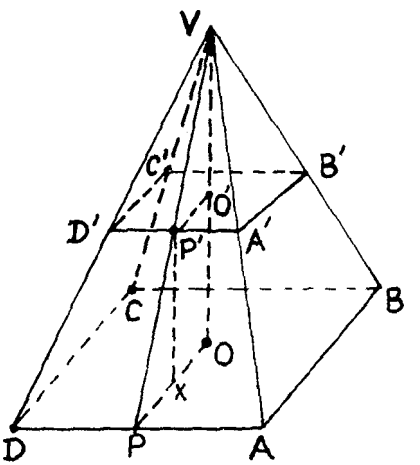


Fig. 10·18

We have, P', P being the mid-points of $A'D'$ and AD and $P'X \perp OP$, where O', O are the centres of the two bases,

$(PP')^2 = (15/2)^2 + (5 - 5/2)^2$,

as $PX = OP - O'P' = 5 - (\frac{5}{2})$.

$\therefore PP' = 5\sqrt{5/2} = A'L$, where seg $A'L \perp AD$.

$\therefore$ Similarly $[A'A]^2 = AL^2 + [A'L]^2$

$= \left(5 - \frac{5}{2}\right)^2 + \left(\frac{125}{2}\right)$,

as $AL = AP - A'P$.

$\therefore A'A = \frac{5\sqrt{11}}{2}$.

$\therefore$ lateral surface $= 5\sqrt{5/2} \times \frac{1}{2}(40 + 20)$.

$= 150\sqrt{5/2}$ sq. cms. $= 75\sqrt{10}$ sq. cms.

Ex. 7 : If from a cone whose base is 100 sq. cms. and height 15 cms. another cone is cut off by a plane parallel to the base and enclosing a sectional area of 36 sq. cms., calculate the height and the volume of the remaining solid.

If h_1 and h_2 are the heights of the two cones, we have

$\frac{100}{h_2^2} = \frac{36}{h_1^2}$; but $h_2 = 15$; $\therefore h_1 = 9$.

$\therefore$ height of the frustum = 6 cms.

Volume of the frustum $= \frac{6}{3}\left[100 + \sqrt{100 \times 36} + 36\right]$

$= 392$ sq. cms.

Ex. 8 : A right circular cone is cut by two planes parallel to the base and trisecting the height. Compare the volumes of the three pieces.

Let the height of the cone be $3h$ so that the height of each piece is h; let v_1, v_2, v_3 be the volumes of the three pieces.

Since the volumes of similar right circular cones are in the triplicate ratios of their linear dimensions, we have

$$\frac{V_1}{h^3} = \frac{V_1 + V_2}{(2h)^3} = \frac{(V_1 + V_2 + V_3)}{(3h)^3},$$

$$\therefore \frac{V_1}{1} = \frac{V_1 + V_2}{8} = \frac{V_1 + V_2 + V_3}{27}$$

$$\therefore \frac{V_1}{1} = \frac{(V_1 + V_2) - V_1}{8 - 1} = \frac{(V_1 + V_2 + V_3) - (V_1 + V_2)}{27 - 8}.$$

$$\therefore \quad V_1 : V_2 : V_3 = 1 : 7 : 19.$$

Ex. 9 : The measure of the vertical angle of a right circular cone is 60°. A point is taken on the curved surface of the cone such that its distance from the vertex is the same as the height of the cone. A plane parallel to the base is drawn through this point. Show that the whole surfaces of the two parts are equal and find the same if the height of the cone be 10 cms.

We have, the radius of the cone $r_2 = 10 \tan 30° = 10/\sqrt{3}$ cms., and the generator of the cone, $l_2 = 10 \sec 30° = 20/\sqrt{3}$ cms.

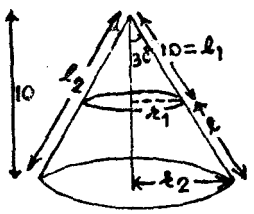


Fig. 10·19

The generator of the upper cone, $l_1 = 10$ cms., and its height $l_1 \cos 30° = 5\sqrt{3}$ cms. and radius $r_1 = l_1 \sin 30° = 5$ cms.

$\therefore$ the whole surface of the upper cone

$= \pi r_1^2 + \pi r_1 l_1$

$= 5\pi(5 + 10) = 75\pi$ sq. cms.

The whole surface of the frustum

$= \pi r_2^2 + \pi(r_2 + r_1)\, l$

$$= \pi\left\{\left[\frac{10}{\sqrt{3}}\right]^2 + 5^2 + \left[\frac{10}{\sqrt{3}} + 5\right]\left[\frac{20}{\sqrt{3}} - 10\right]\right\} = 75\pi \text{ sq. cms.}$$

Hence the result.

Exercise 10

(1) Fill in the blanks in the following :

(i) Each lateral face of a pyramid is a.....region

(ii) In a .....pyramid each vertex of the base is equidistant from the vetrex of the pyramid.

M.P.I. 13

(iii) If two triangular pyramids have congruent bases, their volumes are proportional to their .....

(iv) The area of a section of a pyramid by a plane parallel to its base is proportional to the.....of its distance from the vertex of the pyramid.

(v) A prism and a pyramid have a common base and the height of the prism is one half that of a pyramid. Hence their volumes are as..... : .....

(vi) If a cone and a cylinder have bases of equal area and have equal altitudes then the volume of the cylinder is.....times the volume of the cone.

(2) State whether the following statements are true or false.

(i) If the lateral faces of a pyramid are congruent isosceles triangles, then the pyramid is a regular pyramid.

(ii) If the lateral faces of a pyramid are isosceles triangles, then the pyramid is a right pyramid.

(iii) Every right pyramid is a regular pyramid.

(iv) A pyramid does not have a diagonal.

(v) A right pyramid on a rectangular base has a constant slant height for all its lateral faces.

(3) Choose the correct answer from the given alternatives.

(i) The total surface area of a right circular cylinder of base radius a and height h is
(a) πa^2h, (b) $2\pi a(a+h)$, (c) $\pi a^2 + \pi ah$, (d) $\pi a^2 + 2\pi ah$.

(ii) The radius of the base and the height of a right circular cone are each increased by 10%. Hence its volume is increased by
(a) 20%, (b) 33·1%, (c) 33%, (d) 25%.

(iii) C_1 and C_2 are two similar right circular cylinders. If the radius of C_2 is 9 cm. and the ratio of the lateral surfaces is 25/36, then the radius of C_1 is
(a) 6 cm., (b) 7·5 cms., (c) 4·5 cms.

(iv) The area of the base of a cylinder is 9 sq. cm. and its height is 5 cm.; then the volume of the cone of the same base area and height is
(a) 45 c. cm., (b) 15 c. cms., (c) 22·5 c. cm., (d) 7·5 c. cm.

(v) The volume of a right circular cone is 50 c. cm. A right circular cylinder of the same height as that of the cone is constructed on the base of the circular cone; then the volume of the cylinder is
(a) 50 c. cm. (b) 100 c. cm. (c) 150 c. cm. (d) 200 c. cm.

(4) Fig 10·20. shows a regular pyramid of height 12 cm. on a square base of side 10 cm. If $A'B'C'D'$ is a section of the pyramid by a plane parallel to the base $ABCD$ and at a distance of 4 cm. from its vertex V, find

(i) $VA'/A'A$,

(ii) $A'B'/AB$,

(iii) area of $\square A'B'C'D'$,

(iv) slant height VL,

(v) total surface area of $[V - ABCD]$,

(vi) volume of the frustum $[A'B'C'D' - ABCD]$.

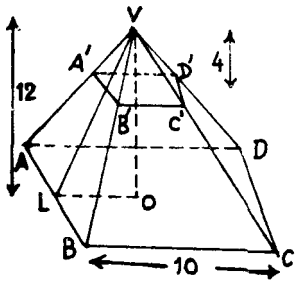


Fig. 10.20

(5) **Find the volume of a regular square pyramid whose lateral faces are equilateral triangles of sides 10 cm.**

(6) **The volume of a pyramid is 500 c. cm. A plane drawn parallel to the base bisects the altitude of the pyramid. Find the volume of the frustum.**

(7) **The base of a pyramid is a square of side 5 cm. If the total surface area of the pyramid is 90 sq. cm. find the height of the pyramid.**

(8) **A right circular cone has altitude 20 cm. and its semi-vertical angle is 30°. Find its volume and lateral surface area.**

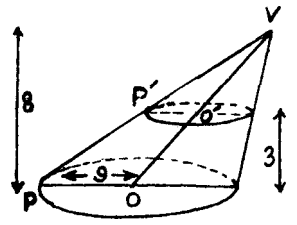


Fig. 10.21

(9) An oblique cone has a circular base with its centre at O and of radius 9 cm., and its altitude is 8 cm. A plane parallel to the base and at a distance 3 cm. from it meets the cone in a circle with O' as its centre. Find the area of this section and also VO/VO'.

[**Hint:** $\dfrac{\text{area of the section}}{\text{area of the base}} = \dfrac{\text{square of the height of the sectional cone}}{\text{square of the height of the given cone}}$].

(10) A right circular conical tent of canvas is to be built to accommodate 77 people under the following conditions. The amount of canvas available is 550 sq. m. and each person must have 2 sq. m. space on the ground. Find the height of the tent so built and also the amount of air available per person. [$\pi = 22/7$].

(11) A circus-tent consists of a cylinderical base surmounted by a right circular conical roof. The height of the cylinderical portion is 9 m. while the highest point of the roof is 24 m. above the ground. The number of persons admitted must be such that each person has 150 c.m. of air space. If the common diameter of the cylindrical base and the conical roof be 60 m. find the number of persons that could be admitted. [$\pi = 22/7$.]

[**Hint:** Volume of the cylindrical portion
$= \pi (30^2)(9)$ cu.m.
Volume of the conical section
$= \frac{1}{3} \pi (30^2)(15)$ cu.m.
Volume of the interior of the tent
$= \pi (30^2)(9+5)$. cu.m.].

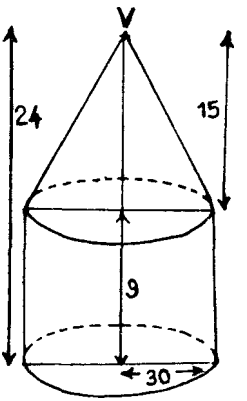


Fig. 10.22

(12) A canvas tent is in the shape of a right regular hexagonal based prism surmounted by a regular pyramid. If the side of the hexagon be $2\sqrt{3}$ m., the height of the prism 3m. and the height of the pyramidal portion of the tent be 4 m. find the quantity of the canvas used in the preparation of the tent.

(13) Find the edge of the greatest cube that can be cut out from a right circular cone, 12 cm. high and with 8 cm. as the diameter of the base so that the base of the cube lies in the base of the cone.

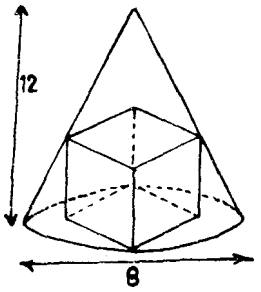


Fig. 10.23

[**Hint** : Let the length of an edge of the cube $= x$ cm.

$\therefore$ diagonal of the face of the cube $= x\sqrt{2}$ cm.

$\therefore$ by similar triangles, we get

$\frac{12-x}{x\sqrt{2}} = \frac{12}{8}$].

(14) A regular pyramid stands on an equilateral triangle of side 12 cms. If its height is half of its slant height, find its volume and the slant edge.

(15) The radii of two circular sections of a right circular cone are 10 cms. and 7 cms, the distance between the two sections being 3 cms. Find the volume of the frustum.

(16) The top of a tub is a rectangle of sides of length 1·8 m. and 1 m. and its bottom is a rectangle whose sides are parallel to those of the top and are of lengths 1·2 m. and 0·6 m. If the depth of the tub be 1 metre, find the volume of the tub.

(17) The diameters of the top and bottom of a bucket are 28 cm. and 18 cm. respectively. If the bucket is 30 cm. deep, calculate the volume of water when the bucket is full.

(18) A piece of paper in the form of a sector of a circle of radius 10 cms. with its angle of measure 216° , is rolled into a right circular cone. Find the volume of the cone.

(19) An equilateral $\triangle ABC$ is inscribed in the base circle of the right circular cone $(V\text{-}ABC)$. Find the ratio of the volumes of the cone $(V\text{-}ABC)$ and the pyramid $(V\text{-}ABC)$.

(20) Water is poured into a conical wine glass till half the inner surface of the glass is below the surface of the wine. Show that the glass is a little more than 35% full.

(21) The number showing the volume of a regular tetrahedron is 1/3 times the number showing its total surface area. Prove that its volume is $8\sqrt{3}$ cub. units.

*(22) A pyramid is divided by three planes parallel to the base into four parts of equal volume. Prove that the heights of these parts are

$$1,\ (2^{1/3}-1)\ (3^{1/3}-2^{1/3}) \text{ and } (4^{1/3}-3^{1/3}) \text{ times } h/4^{1/3}$$

respectively starting from the vertex, where h is the height of the pyramid.

*(23) A', B', C' are points in the edges seg VA, seg VB, seg VC of the pyramid $[V-ABC]$. Prove that

$$\frac{\text{Vol}\,[V-A'B'C']}{\text{Vol}\,[V-ABC]} = \frac{VA' \times VB' \times VC'}{VA \times VB \times VC}.$$

(24) A right circular cone is cut by a plane parallel to the base and the upper portion is removed. If the curved surface of the frustum is 8/9 times of the curved surface of the whole cone, find the ratio of the segments into which the altitude of the cone is divided by the plane.

(25) From a cylinder of height 12 cms. and radius 5 cms. a conical cavity of the same height and base is hollowed out. Find the volume and whole surface of the remaining solid.

Answers

(1) (i) triangular, (ii) right, (iii) heights, (iv) square, (v) 3:2, (vi) three.

(2) (i) true, (ii) true, (iii) false, (iv) true, (v) false.

(3) (i) (b,) (ii) (b,) (iii) (b,) (iv) (b,) (v) (c).

(4) (i) 1/2, (ii) 1/3, (iii) 100/9 sq. cm., (iv) 13 cm. (v) 360 sq. cm. (vi) $385\frac{5}{27}$ c. cm.

(5) $(500\sqrt{2})/3$ c. cm. (6) 437·5 c. cm. (7) 6 cm.

(8) $8000\pi/9$ c. cm.; $800\pi/3$ sq. cm. (9) $2025\pi/64$ sq. cm., 8/5. (10) 24 m., 16 c. m.

(11) 264. (12) $66\sqrt{3}$ sq. m. (13) $24/(2+3\sqrt{2})$ cm.

(14) slant edge $2\sqrt{13}$ cm., Volume $24\sqrt{3}$. c. cm.

(15) 219 cub. cm. (16) 1·22 c. m. (17) 4030π c. cm.

(18) 96π c. cm. (19) $3\sqrt{3}/2\pi$. (24) 1:2, (25) 200π c. cm. 210π. sq. cm.

□ □

CHAPTER 11

The Sphere

(1) Sphere. Definition : *A sphere is the set of all points in space such that the distance of each of its points from a given point is the same.*

The given fixed point O, say, is called the **centre** of the sphere and the fixed distance r, say, is called a radius of the sphere. The sphere may be denoted by (O, r). The set of all points in space at a distance $< r$, from O, will be called the interior of the sphere; and the set of all points at a distance $> r$, from O, will be called the exterior of the sphere.

If P be a point on the above sphere, then $OP = r$; similarly if $OP = r$. then P will be a point on the above sphere.

It will be seen that the phrase 'radius' of a sphere connotes two meanings : (i) a positive real number denoting the constant distance and (ii) a segment joining the centre of the sphere to a point on the sphere.

Thus r ($= OP$) is a radius; and seg OP is also called a radius. Thus any segment, whose one end is the centre O of the sphere and the other end P, say, is a point on the sphere, is called a radius of the sphere.

(2) Line and a Sphere : If a line l, say, intersects a sphere, the intersection set contains at the most two points of the sphere. If the intersection set consists of only one point, the line is said to touch the sphere and it is then called a **tangent** to the sphere. If the intersection set consists of two points say P and Q then seg PQ is called a chord of the sphere. The line itself which intersects the sphere in two distinct points is called a **secant** of the sphere.

A diameter of a sphere is a chord of the sphere passing through the centre of the sphere; thus if A and B are two points on the sphere (O, r) such that we have $A—O—B$ i.e., O lies in between A and B then seg AB is a diameter of the sphere and $AB = 2r$. It is thus clear that the phrase 'diameter of a sphere' has also two meanings : (i) a positive real number and (ii) a segment. It will be always clear from the context in which particular sense the terms 'diameter' and 'radius' are used.

It is also clear that

(i) all diameters of a sphere are of the same length viz. $2r$, where r is the (length of a) radius;

(ii) all diameters are concurrent in the centre O of the sphere.

It will also be clear that a line

(i) may not intersect a sphere.

(ii) may intersect it in two points in which case it is called a secant of the sphere,

(iii) may intersect it in only one point, in which case it is a tangent to the sphere. The intersection set consisting of the single point, in this case, is called the point of contact of the tangent to the sphere.

(3) Plane and a Sphere. We consider now the possible relationships between a plane $\propto$ and a sphere (O, r). The intersection set of a plane $\propto$ and a sphere may be an empty, i.e. a null, set. Another possibility is that this intersection set is not a null set. We consider this latter possibility and prove the following theorem.

Theorem. The intersection set of a sphere and a plane, when not empty, is a circle.

Consider a sphere $S (O, r)$ and a plane $\propto$. Let $C = S \cap \propto$.

We then prove that C is a circle. We take any point P belonging to C.

Let OL be $\perp$ to the plane $\propto$ at L. Join OP and LP. (See Fig 11·1)

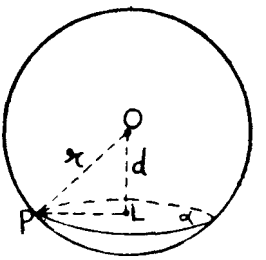


Fig. 11.1

Now as seg $OL \perp$ plane $\propto$,

$\therefore$ seg $OL \perp$ to seg LP, as seg LP lies in $\propto$.

$\therefore$ from the right-angled triangle OLP

$$LP^2 = OP^2 - OL^2.$$

If OL, which is the distance of the plane $\propto$ from the centre O of the sphere, be denoted by d we get

$$LP = \sqrt{r^2 - d^2}, \text{ since } OP = r.$$

Now L is a fixed point on the plane $\propto$ and P is a point, on the plane $\propto$ ($\because$ P belongs to $S \cap \propto$), at a constant distance $\sqrt{r^2 - d^2}$ from L.

$\therefore$ P lies on the circle $(L, \sqrt{r^2 - d^2})$ in the plane $\propto$.

Conversely any point on this circle, in the plane $\propto$, can be shown to lie on the sphere S. For, if P lies on the circle $(L, \sqrt{r^2-d^2})$ in the plane $\propto$, then from a right-angled $\triangle OLP$, we have

$$OP^2 = OL^2 + LP^2 = d^2 + (r^2 - d^2) = r^2.$$

$\therefore \quad OP = r, \quad \therefore \quad P$ is a point on the sphere S. Thus the intersection of the sphere S and the plane $\propto$ is a circle, of radius $\sqrt{r^2-d^2}$ where r is a radius of the sphere S and d is the distance of the plane $\propto$ from the centre O of the sphere.

Hence sections of a sphere of radius r by planes at a distance d from its centre are circles of the same radius $\sqrt{r^2-d^2}$ and hence are congruent circles.

The converse of this result is also true. For if d_1 and d_2 are the distances of two congruent circular sections of the same radius a, of a sphere of radius r, we have

$$r^2 - d_1^2 = a^2 = r^2 - d_2^2, \quad \text{and} \quad \therefore \quad d_1 = d_2.$$

Cor (i). If $0 < d < r$, the sphere S intersects the plane $\propto$ in a circle of radius $\sqrt{r^2-d^2}$ which is less than the radius of the sphere.

Such a circle on a sphere is called a **small circle.**

Cor (ii). If $d = 0$, the sphere intersects the plane $\propto$ in a circle whose radius is $\sqrt{r^2-0^2} = r$, i. e. the same as the radius of the sphere. *Such a circle on a sphere is called a* **great circle.** Thus sections of a sphere by planes passing through its centre are all great circles and are congruent circles of the same radius as that of the sphere.

Cor (iii). If $d = r$, the intersection of the sphere S and the plane $\propto$ is a point circle, i.e.. a circle of zero radius. In this case *the plane* $\propto$ *is said to touch the sphere and is called its* **tangent plane**. If P is the single point in the intersection set of the sphere and the plane, then P is called the **point of contact** of the tangent plane and seg OP which is $\perp$ plane $\propto$ at P is called the radius to the point of contact. (See Fig. 11·2)

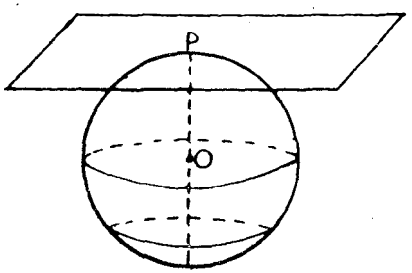


Fig. 11.2

Cor. (iv). If $d > r$, the intersection of the sphere and the plane does not consist of a set of real points and, to generalise the concept, we say that the intersection is an imaginary circle.

(4) Sphere as a Surface of Revolution. The significance of the result proved above in Cor. (ii) is that we can now give an alternative definition of a sphere.

If we take a plane through a fixed diameter of a sphere and revolve it completely about that diameter, its intersection with the sphere is always a great circle. Hence we can define a sphere as a surface obtained by the revolution of the perimeter of a semicircle about its diameter.

(5) Some Definitions

(i) Frustum. *The portion of the volume of a sphere included between two parallel planes is called a spherical segment of two bases or a frustum of the sphere.*

The two plane faces are known as the **bases** of the frustum and the perpendicular distance between them is called the **height** of the frustum.

(ii) Zone. *The curved surface of a frustum of a sphere is called a zone of the sphere.*

(iii) Spherical Cap *The portion of the volume of a sphere cut off by a plane is called a spherical cap. It is also called a spherical segment of one base.*

Any plane intersecting a sphere divides it into spherical caps of heights $(a-h)$ and $(a+h)$ where h is the distance of the plane from the centre of the sphere and a is its radius.

(iv) Hemisphere. *Any plane through the centre of a sphere divides it into two equal parts called hemispheres.* A hemisphere is a spherical cap whose height is equal to a radius of the sphere.

(6) Volume of a sphere.

Theorem. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Let $x^2+y^2=r^2$ be a circle and let seg AB be a diameter where $A=(-r, 0)$, $B=(r, 0)$. Let P be a point on the circle in the first or second quadrant and P' be a point on it very near to P.

Let $P \equiv (x, y)$ and $P' \equiv (x+\delta x, y+\delta y)$. Let seg PL and seg $P'L'$ be $\perp$ seg AB. (See Fig. 11·3).

The revolution of the semi-circle about the diameter AB generates a sphere of radius r and simultaneously the area $LPP'L'$ (bounded by the X-axis, the ordinates of P and P' and the arc PP') will generate a small frustum of the sphere of volume δV, say.

If this frustum is very thin, so that δx is very small, it can be regarded as a thin cylindrical plate and we get $\delta V = \pi y^2 \delta x$ to the first order of small quantities.

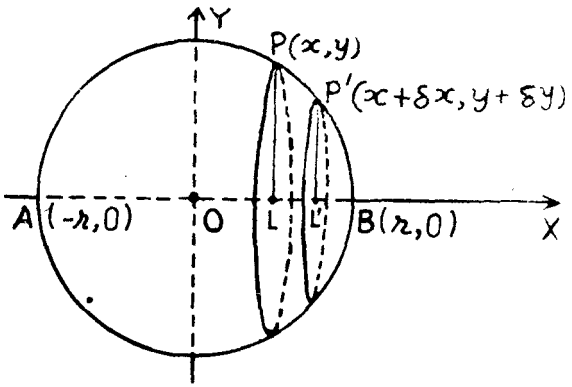


Fig. 11.3

Dividing the sphere into a large number of such thin cylindrical slices by planes perpendicular to the X-axis and denoting the volume of the sphere by V, we get

$$\begin{aligned} V &= \Sigma \delta V = \Sigma \pi y^2 \, \delta x \\ &= \Sigma \pi (r^2 - x^2) \, \delta x, \quad \because \quad x^2 + y^2 = r^2 \\ &= \pi \int_{-r}^{r} (r^2 - x^2) \, dx = \pi \left[r^2 x - x^3/3 \right]_{-r}^{r} \\ &= \pi \left[(r^3 - r^3/3) - (-r^3 + r^3/3) \right] \\ &= \tfrac{4}{3} \pi r^3. \end{aligned}$$

Alternative Method : We can alternatively find the volume of a sphere by using Cavalieri's principle as follows :

Consider a right circular cylinder of base radius r and height $2r$ together with an inscribed sphere of radius r (which touches the bases of the cylinder at their centres and also touches the lateral surface of the cylinder along a great circle) and the two right circular cones with the bases of the cylinder as their bases and a common vertex at the centre O of the inscribed sphere (see Fig. 11·4 on page 203).

The area of a section of the sphere by a plane at a distance x [$= OL < r$] from the centre O is $= \pi (r^2 - x^2)$ and the area of the section of the solid consisting of the cylinder minus the double cone, being a ring whose inner and outer radii are $LP = OL = x$ and LQ ($= r$), is also $\pi r^2 - \pi x^2 = \pi (r^2 - x^2)$.

Hence the sphere and the solid consisting of the cylinder minus the double cone are Cavalieri bodies and thus have equal volumes.

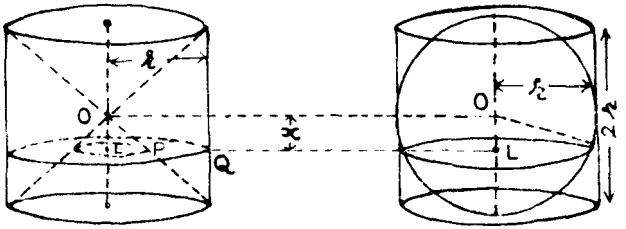


Fig. 11·4

Hence the volume of the sphere

$$= \text{vol. of the cylinder} - \text{vol. of the double cone}$$
$$= \pi r^2 \times 2r - 2 \times \tfrac{1}{3}\pi r^3 = \tfrac{4}{3}\pi r^3.$$

(7) Comparison between Volumes of two Spheres. From the expression for the volume of a sphere, we can now deduce the following result.

Theorem. The volumes of two spheres are in the ratio of the cubes of their radii.

For if V_1 and V_2 are the volumes of two spheres with radii r_1 and r_2 respectively, we have

$$V_1 = \frac{4}{3}\pi {r_1}^3, \quad V_2 = \frac{4}{3}\pi {r^3}_2,$$

and $$\therefore \quad \frac{V_1}{V_2} = \frac{\frac{4}{3}\pi {r_1}^3}{\frac{4}{3}\pi {r_2}^3} = \frac{{r_1}^3}{{r_2}^3}.$$

(8) Volume of a spherical segment of two bases (frustum).

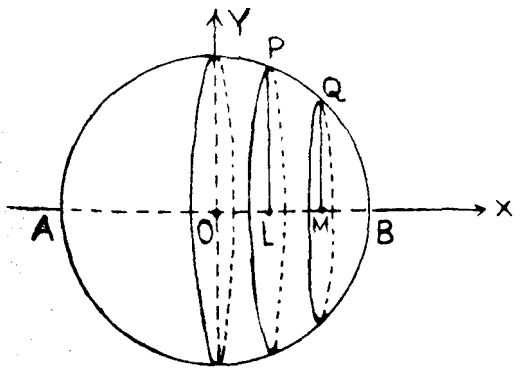


Fig. 11.5

Theorem. The volume of a spherical segment of two bases of radii r_1 and r_2 and of height h is $\frac{\pi h}{6}(3{r_2}^2 + 3{r_1}^2 + h^2)$.

Let h_1 and h_2 be the distances of the two parallel bases of the segment from the centre O of the sphere. We consider the segment a

being generated by the area $LPQM$ where $OL = h_1$, and $OM = h_2$ so that $h_2 - h_1 = h$, (see Fig. 11·5).

The volume of the spherical segment of the two bases which are the two circles with L and M as centres and LP and MQ as radii and lie in the planes $\perp$ AOB at L and M respectively is

$$= \int_{h_1}^{h_2} \pi (r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{h_1}^{h_2}$$

$$= \pi \left[r^2 (h_2 - h_1) - \tfrac{1}{3} (h_2^3 - h_1^3) \right]$$

$$= \tfrac{1}{3} \pi h \left[3r^2 - (h_2^2 + h_1 h_2 + h_1^2) \right].$$

But $LP = r_1$ and $MQ = r_2$;

$\therefore$ we have $OP^2 = OL^2 + LP^2$,

i. e. $r^2 = h_1^2 + r_1^2$; similarly $r^2 = h_2^2 + r_2^2$.

$\therefore$ the volume of the segment

$$= \frac{\pi h}{6} \left[6r^2 - 2 (h_2^2 + h_1 h_2 + h_1^2) \right]$$

$$= \frac{\pi h}{6} \left[6r^2 + (h_2 - h_1)^2 - 3h_2^2 - 3h_1^2 \right]$$

$$= \frac{\pi h}{6} \left[h^2 + 3 (r^2 - h_2^2) + 3 (r^2 - h_1^2) \right]$$

$$= \frac{\pi h}{6} (3r_2^2 + 3r_1^2 + h^2)$$

where r_1 and r_2 are the radii of the circular bases.

(9) Volume of a Spherical cap. (See Fig 11·5).

The volume of a spherical cap of base-radius a and height h is $\frac{\pi h^2}{3} (3r - h)$ and is $\frac{\pi h}{6} (3a^2 + h^2)$, where r is the radius of the sphere.

Let the cap be generated by the rotation of the area $LPQB$ where the height of the cap viz. $LB = h$ so that OL now $= r - h$. Let $LP = a$ be a base-radius of the cap. Hence the volume of the cap.

$$= \int_{r-h}^{r} \pi (r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left[r^2 h - \frac{1}{3} \{ r^3 - (r-h)^3 \} \right]$$

$$= \frac{\pi h}{3} [3r^2 - \{ r^2 + r(r-h) + (r-h)^2 \}]$$

$$= \frac{\pi h}{3} [3rh - h^2] = \frac{\pi h^2}{3} (3r - h) \quad .. \quad .. \quad .. \quad (i)$$

$$= \frac{\pi h}{6} (6rh - 2h^2)$$

$$= \frac{\pi h}{6} (3a^2 + 3h^2 - 2h^2), \text{ since } r^2 = (r-h)^2 + a^2, \text{ i. e. } 2rh = a^2 + h^2,$$

$$= \frac{\pi h}{6} (3a^2 + h^2).$$

(10) Relation between the Volume and Surface of a sphere.

Theorem. If r is the radius of a sphere of volume V and surface area S, then $V = \frac{1}{3} rS$.

We divide the surface of the sphere into small elementary parts. Let δS be a typical one of these. Let δV be the volume bounded by δS and the radii of the sphere through the points in its boundary. If δS be very small, so that it may be regarded as a small plane area, δV becomes the volume of a cone of base area δS and height r.

$$\therefore \delta V = \frac{1}{3} r \delta S.$$

$$\therefore \quad V = \Sigma \delta V = \Sigma \frac{1}{3} r \delta S = \frac{1}{3} r \Sigma \delta S = \frac{1}{3} rS.$$

Cor. The surface S of a sphere of radius r is given by $S = 4\pi r^2$.

For, $S = \frac{3V}{r} = \frac{3}{r} \times \frac{4}{3} \pi r^3 = 4\pi r^2$.

Illustrative Examples

Ex. 1. $[V - ABC]$ is a regular tetrahedron of edge 6 cms. Find the volume of its (i) circumscribing sphere, (ii) inscribed sphere.

Let VO be an altitude of the tetrahedron and G, lying in VO, be its centroid so that $VG = 3\,GO$.

Further O is the circumcentre of the equilateral triangle ABC.

$\therefore \quad OC = \frac{2}{3} \times$ (median of $\triangle ABC$)

$= \frac{2}{3} \times \left(6 \times \frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$ cms.

$\therefore \quad VO^2 = VC^2 - OC^2 = 36 - 12 = 24.$

$\therefore \quad VO = 2\sqrt{6}$, cms., and $VG = \frac{3\sqrt{6}}{2}$ cms. and $GO = \frac{\sqrt{6}}{2}$ cms.

Now the common centre of the inscribed and circumscribing spheres is G and their radii are seg GO and seg GV respectively.

$\therefore$ vol. of the inscribed sphere

$$= \frac{4}{3}\pi \times \frac{6\sqrt{6}}{8} = \pi\sqrt{6} \text{ c. cms.},$$

and vol. of the circumscribing sphere

$$= \frac{4}{3}\pi \times \frac{27}{8} \times 6\sqrt{6} = 27\pi\sqrt{6} \text{ c. cms.}$$

Ex. 2. A plate in the form of a regular polygon of n sides, each side equal to b, rests on the inside of a spherical bowl of internal radius a. Prove that the distance of the plate from the centre of the bowl is

$$\left(a^2 - \frac{b^2}{4}\operatorname{cosec}^2\frac{\pi}{n}\right)^{1/2}.$$

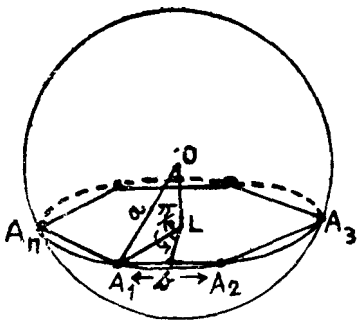


Fig. 11.6

The circum-radius of a regular n-gon, with centre L, is

$$\frac{b}{2}\operatorname{cosec}\frac{\pi}{n}.$$

If O be the centre of the bowl, then seg OL is perpendicular to the plane of the n-gon.

$\therefore$ the required distance OL is given by

$$OL^2 + \frac{b^2}{4}\operatorname{cosec}^2\frac{\pi}{n} = a^2.$$

$$\therefore \quad OL = \left(a^2 - \frac{b^2}{4}\operatorname{cosec}^2\frac{\pi}{n}\right)^{1/2}.$$

Ex. 3. A sphere of diameter 8 cms. is dropped into a cylindrical vessel partly filled with water. If the sphere is completely sub-merged, by how much will the surface of the water be raised if the diameter of the vessel be 12 cms. and no water has flowed out ?

The volume of the sphere $= \frac{4}{3}\pi \times 4^3 = \frac{256\pi}{3}$ c. cms.

Area of the base of the cylindrical vessel

$= \pi \times 6^2 = 36\pi$ sq. cms.

$\therefore$ rise in the water level $= \frac{256\pi}{3} \div 36\pi = \frac{64}{27}$ cms.

Ex. 4. A sphere of radius 5 is placed on a cylindrical vessel of base-radius 4 Find the volume of the part of the sphere which will be inside the sphere.

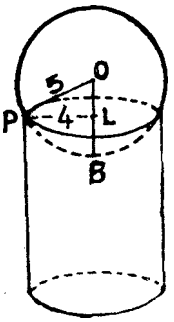


Fig. 11.7

Let O be the centre of the sphere. The part of the sphere within the cylindrical vessel is a spherical cap. The centre of the base of the cap is L and its base-radius is $LP = 4$, which is the radius of the base of the cylinder. The height of the spherical cap is.

$$LB = OB - OL = 5 - OL,$$

where $$OL = \sqrt{OP^2 - LP^2} = \sqrt{5^2 - 4^2} = 3.$$

$\therefore$ the height of the cap, LB, say h, $= 2$.

Now the volume of a spherical cap of base radius a and height h is $\frac{\pi h}{6}(3a^2 + h^2)$.

$\therefore$ the required volume is

$$\frac{\pi}{6}(2)[3(4^2) + 2^2] = \frac{\pi}{3}[52] = \frac{52\pi}{3}.$$

Ex. 5. The diameter of a spherical ball is 12 cms. It floats in water such that its highest point is 2 cms. above the surface of the water. Find the ratio of the volume of the portion in water to the volume of the ball.

Volume of the ball $= \frac{4}{3}\pi \times (6)^3 = 288\pi$ c. cms.

The height of the segment which is under water $= (12 - 2) = 10$ cms.

$\therefore$ volume of this segment $= \frac{\pi \times 10^2}{3}(18 - 10) = \frac{800\pi}{3}$ c. cms.

$\therefore$ required ratio $= \frac{800\pi}{3} \div 288\pi = \frac{25}{27}$.

EXERCISE 11

(1) A sphere is inscribed in a right circular cylinder. Prove that their volumes are proportional to their total surfaces.

(2) A right circular cone circumscribes a sphere. Prove that their volumes are proportional to their total surfaces.

(3) A right circular cone, a hemisphere and a right circular cylinder have a common base and a common height. Prove that their total surfaces are as $(1 + \sqrt{2}) : 3 : 4$.

(4) Find the volume of a spherical cap of height 3 cms. the radius of the sphere being 6 cms.

(5) Find the volume of a spherical cap of height 2 cms. and base-radius 4 cms.

(6) A cube is inscribed in a sphere. If V_1 and V_2 be the volumes of the cube and the sphere, prove that $V_1 : V_2 = 2 : (\pi\sqrt{3})$.

(7) A regular tetrahedron is inscribed in a sphere of radius r. Find the radius of the circular section of each its of faces.

(8) The altitude of a right circular cone is equal to half the radius of the base. Find the radius of the sphere whose volume is equal to that of the cone, if the radius of the cone be 10 cms.

(9) A sphere is cut by a plane which divides the perpendicular diameter in the ratio $k : 1$. If V_1, V_2 are the volumes of the two spherical caps find $V_1 : V_2$.

*(10) A hole in the shape of a double cone of semi-vertical angle of measure 60° is drilled through a sphere so that the common vertex of the double cone lies at the centre of the sphere. Show that the volume of the remaining portion is equal to that of the portion removed.

*(11) A cylindrical hole of 3 cms. radius is drilled into a sphere whose radius is 5 cms., the axis of the cylindrical hole passing through the centre of the sphere. Find the volume of the remaining portion of the sphere.

*(12) To the ends of a cylinder whose length is equal to its diameter are applied hemispheres of the same diameter as this cylinder. If the volume of the whole solid thus formed is 97.02 cub. cms. find the surface area of this solid. (Take $\pi = 22/7$.)

*(13) A hemisphere is resting on the ground and a right cone lie on the opposite sides of a common base of 8 cms. diameter and the measure of the semi-vertical angle of the cone is 45°. If a cylinder should circumscribe them in position, find how much additional space is enclosed thereby.

*(14) A sphere 15 cms. in diameter is divided into three parts of equal height by two parallel planes. Find the volume of each part.

(15) A conical glass whose depth is 12 cms. and the diameter of whose mouth is 10 cms. is filled with water. A sphere of 8 cms. diameter and of greater specific gravity than that of water is put into the glass. How much water will run over out of the glass ?

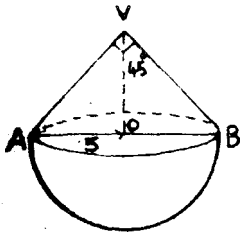
(16) A leaden sphere, one centimeter in diameter, is beaten into a circular sheet of uniform thickness .01 cm. Find the radius of the sheet.

(17) The radius of a hemisphere is 7 cms. Find the altitude of a right circular cone on the same base and having the same volume.

(18) A solid sphere of metal with diameter 60 cms. was melted and cast into circular coins of diameter 3 cms. and thickness 2 mm. Find the number of coins cast.

(19) A solid metal cone with the radius of the base 9 cms. and altitude 16 cms. is melted and solid spherical balls of diameter 6 cms. are made out of it. Find the number of balls made.

*(20) The curved surface of a solid conical vessel of copper is $47\frac{1}{7}$ sq. cms. and the area of its base is $28\frac{4}{7}$ sq. cms. The solid is melted into solid spherical balls each of diameter 0·2 cms. Find the number of balls obtained. (Take $\pi = 22/7$.)



(21) A solid (Fig. 11.8) is composed of a right circular cone and a hemisphere on opposite sides of a common circular base of radius 5 cms. The measure of the semi-vertical angle of the cone is 45°. Find the volume of the solid figure.

Fig. 11.8

22) A sphere is inscribed in a right circular cone of height 9 cms. (Fig. 11.9). The measure of the semi-vertical angle of the cone is 30°. Find the volume of the exterior of the sphere which is interior to the cone.

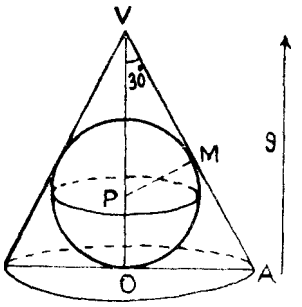


Fig. 11.9

(23) Fill in the blanks in the following :

(i) The radius of a sphere of volume 288π cub. cms. is.....cms.

(ii) The ratio of the surface area of a sphere to the surface area of a rectangular prism circumscribing it is... : .... ($\pi = 22/7$).

(iii) A right circular cone and a hemisphere have the same base and the height of the cone equals the radius of the hemisphere. Hence their volumes are as ... : ... and their curved surfaces are as ... : ... .

(iv) The volume of a sphere is ..... times the volume of its circumbscribing right circular cylinder.

(v) If a diameter of a base and the height of a right circular cylinder are each equal to a diameter of a sphere, then their volumes are as ... : .... .

(24) Complete the following sentences by choosing the correct answer from the given alternatives viz.

(a) square, (b) squares, (c) cube, (d) cubes.

(i) The volume of a sphere is proportional to the ..... of its radius.

(ii) The surface areas of spheres are proportional to the.....of their radii.

(iii) The volumes of spheres are proportional to the ..... of their radii.

(25) Answer the following :

(i) If the radii of two spheres are 3 cms. and 4 cms. respectively, which of the following gives the radius of the sphere whose surface area is equal to the sum of their surface areas ?

(a) 100π cms, (b) 5 cms, (c) 10 cms, (d) none of these.

(ii) If eight spherical shots each of radius 2 cms, are melted and a new spherical shot is formed from them, then the radius of the new shot is

(a) 8 cms., (b) 16 cms., (c) 4 cms., (d) 2 cms.

(iii) The radii of the base of a right circular cone and of a sphere are 4 cms. each. The height of the cone is 4 cms. Hence the ratio of the volumes of the sphere and the cone is

(a) 1, (b) 2, (c) 3, (d) 4.

(iv) The volume of a sphere is 25 c. cms. Hence its surface area is

(a) 15 sq. cms., (b) 125/3 sq. cms., (c) 125 sq. cms. (d) none of these.

ANSWERS

(4) $45\ \pi$ c. cms. (5) $52\pi/3$ c. cms. (7) $4r/3\sqrt{2}$. (8) 5 cms.

(9) $k^3 (k + 3) / (3k + 1)$. (11) $256\pi/3$ cub. cms. (12) 110.88 sq. cms.

(13) 64π cub. cms. (14) $875\pi/6$ cub. cms.; $1625\pi/6$ cub. cms.; $875\pi/6$ cub. cms.

(15) $25088\pi/375$ cub. cms. (16) $5\sqrt{(2/3)}$ cms. (17) 14 cms. (18) 80,000.

(19) 12. (20) 9,000. (21) 125π c. cms. (22) 45π c. cms.

(23) (i) 6, (ii) 11 : 21, (iii) 1 : 2, $1 : \sqrt{2}$, (iv) 2/3, (v) 3 : 2.

(24) (i) cube, (ii) squares, (iii) cubes.

(25) (i) b, (ii) c, (iii) d, (iv) d.

◇ ◇ ◇

CHAPTER 12

Frequency Distributions

(1) What is Statistics ? The word ' statistics ' has several meanings.

(i) The word ' **statistics** ' **as a plural noun** is used to describe collections of numerical data such as employment statistics, accident statistics, statistics of population of a country, of males and females, of refugees, of births and deaths, of income and expenditure, of imports etc. It is in this sense that this word is generally used by a layman or by newspapers.

(ii) The word ' **statistics** ' **as a singular noun**, is used to describe a branch of applied mathematics whose purpose is to provide methods of dealing with collections of data and extracting information from them in a compact form by tabulating, summarising and analysing the numerical data or a set of observations.

The various methods used are known as statistical methods and the person using them is known as a statistician. Thus, a statistician is concerned with analysing and interpreting the data and drawing valid worthwhile conclusions from the same. It is in this second sense that we are writing this course in statistics.

(iii) The word ' **statistics** ' is used in a specialized sense by statisticians to describe various numerical items which are produced by applying statistics (in the second sense) to statistics (in the first sense). Thus, the arithmetic mean, the standard deviation etc. are all statistics in this specialized third sense.

(2) Population and Individual.

Definition : The word population, in statistics, is defined in a wider sense as any collection of objects taken for a statistical enquiry regarding a certain property. Each member (observation) of the collection is called an individual. The population is also known as the **universe.**

(3) Variates and Attributes :—Each individual in the population is observed with respect to a certain character or characterisitic and a measurement of this observation is taken. For example, suppose we are interested in the performance, in the subject of Mathematics, of all the students appearing for the F. Y. examination. In this case, the population consists of all the student examinees and every student is an individual in this population. The characteristic examined is the performance of each student in the subject of Mathematics and this may be measured by observing the number of marks gained by him. Such a character is quantitative. On the other hand we may be interested in the number of students offering a language or Mathematics or Economics as their honours subject at their B. A. examination. In this case, students offering any of these subjects for the B. A. examination will form the population, the various students being the individuals. The character observed is the subject offered; such a character is qualitative. Thus the numerical data in Statistics may arise in two ways according to the nature of the characteristic observed. When a character observed in each individual, like the marks obtained, the age or the height of the individual, can be expressed in numbers, it is **quantitative** and is known as a **variable or a variate**; when the character observed is not expressible in a numerical form, it is **qualitative and is known as an attribute.** This leads us to the following definitions.

Definitions

(i) A variate is a quantitative character, i. e. a property of individulals, in a popopulation, which can be expressed numerically.

Thus to each individual of the population corresponds one and only one value of the variate (variable).

(ii) An attribute is a qualitative character, i.e. a property of individuals, in a population, which can be classified into disjoint non-overlapping classes.

(4) Discrete and Continuous Variables :—If a character observed takes only some isolated values like the number of children that a man has, or the number of letters in a word, or the number of births or deaths in a month, or the number of pages in a book, then the variable measuring the character is a discrete variable. On the other hand, consider the set S of measurements of the heights of students in a certain group; and let x be an element of this set. Then x is a continuous variable. For the height of a student of say, between 135 cms.

and 175 cms., may take any value within the range of variation. Even when we say that the height of a student is, say, 150 cms. it would be correct to the nearest cm. and may be anywhere between 150·5 cms. and 149·5 cms. Other instances of continuous variables are the weight, or the age, in years, of a man or the diameter of a bobbin or the rainfall or humidity in a region and so on. It will thus be seen that measurements are associated with a continuous variable and counts indicate a discrete variable. Thus we may give the following definitions.

Definitions : *A variate is said to be a continuous variate if it can have all possible values in a given interval and it is said to be a discrete variable if it can take only specific isolated values in the interval.*

A data or set of observations is said to be continuous or discrete according as it is described by a continuous or a discrete variable.

(5) **Sample.** Often it is not possible or practicable to record observations for all the individuals of the group forming the population. In such a case observations are recorded for only some of the individuals of the population selected at random.

Definition : *A sample is a set of observations drawn from a population in order to study its characteristics.* If a sample be representative of the whole population, any conclusions drawn from a statistical treatment of the sample would hold reasonably good for the population. This will, of course, depend on the proper selection of the sample. One of the aims of statistics is to draw inferences about the population by a statistical treatment of samples.

Illustrations

(1) The citizens of Bombay are classified according to their (i) mother tongue, (ii) religion, (iii) sex and (iv) family income. State the population and character mentioning in each case whether it is a variate or an attribute.

The population is formed by the citizens of Bombay.

The Mother tongue, religion and sex are the characteristics which are attributes whereas the characteristic " family income " is a variate.

(2) Students in Bombay Colleges are observed for their (i) age, (ii) weight, (iii) height, (iv) sex, and (v) complexion. State in this case the population; mention the characters according as they are variates or attributes. In case of variates mention whether they are continuous or discrete.

The Population is formed by the students in Bombay colleges. Age, weight and height are continuous variates.

Sex and complexion are attributes.

(6) **Collection and Tabulation of Data.** In any statistical investigation, the collection of numerical data is the first and most important matter to be attended to. Often the person investigating the problem will have to collect the data from the actual field of inquiry. He may, for this purpose, issue suitable questionaires to collect the necessary information or he may get it by actual interviews. Thus if one is interested in family income and expenditure on different items in a certain locality, he may address a suitable questionaire on income and items of expenditure to the heads of families. However, a personal interview is preferable since the questionnaire may not evoke adequate response. Another method to collect data is by acutal observations.

Raw Data. Definition. *A set of observations presented as it is, without any kind of classification or analysis, is called raw data.*

Sometimes, the raw data may be available in publications of Government and other public or private organisations.

Such data, however, are often so numerous that the mind can hardly comprehend its significance in the form in which it is exhibited. It, therefore, becomes necessary to tabulate and summarise the data in an easily manageable form. In doing this we often lose sight of details. This is, however, not a serious loss unless we are interested in an individual to which a particular figure or observation refers. Thus, in finding the heights of students in a certain group, we may be interested in the maximum or the minimum height of students in the group; or the average height of the group, or the number of students whose height ranges between 145 cms. and 155 cms. and so on. We would not be interested in the height of each and every member of the group. In fact, in statistics, we are interested in the properties of aggregates and not of individuals.

Frequency. Definition. *The number of observations belonging to a particular group (or having a particular value) is called the frequency of that group (or that value).*

The data in a raw form are not at all effective for a layman. The raw data can, however, be more effectively represented in the form of a table which can reveal it in a compact form. A complete table with the proper headings and sub-headings can reveal the essential features in a clearer perspective. We may add a short note giving the source of the information and the reliability and other points of interest.

When summarizing large masses of raw data, it is often useful to distribute the data into classes or categories and to determine the number of individuals belonging to each class, called the class frequency.

Frequency distribution. A tabular arrangement of data by classes together with the corresponding class frequencies is called a frequeucy distribution or frequeucy table.

We now consider the formation of frequency distributions for various types of data in the next article.

(7) Frequency Distribution of a Discrete Variable (Ungrouped data).

(i) Consider the following figures which give the numbers of members of 90 families residing in a certain housing colony in a Bombay suburb :—

3, 4, 4, 2. 5, 5, 6, 3, 2, 3; 6, 5, 4, 2, 7, 3 4, 8, 2, 3; 5, 4, 10, 9, 4, 8, 6, 5, 11, 2; 9, 10, 3, 4, 7, 8, 9, 2, 5, 4; 4, 2, 6, 9, 7, 5, 3, 4, 12. 5; 8, 3, 9, 10, 2, 7, 4, 6, 7, 5; 2, 2, 5, 6, 6, 8, 7, 3, 5, 7; 10, 7, 9, 8, 4, 4, 7, 5, 5, 5; 9, 6, 3, 4, 6, 7, 3, 2, 12. 5.

The above collected data is difficult to comprehend as it stands. One would naturally be interested in knowing the average size of a family or the largest and smallest families; or the number of families having 5 or more members and so on. Hence, the immediate necessity arises for counting the numbers of families with 2, 3, 4,.... members. We note that the smallest family consists of 2 members and the largest of 12 members. We, therefore, form a table with 11 classes for the eleven values 2, 3, 4,....., 12. We prepare a table with 3 columns; the first column will represent the number of members in a family and the third column will show the numbers of families having the corresponding numbers of members. In the second or the middle column, which is known as the tally sheet column or the score sheet, we take the given values of the variable (viz. the number of members in a family) and for each value place a stroke (tally mark) in the table against the appropriate class. To facilitate counting we arrange the strokes in bunches of five, every fifth stroke being drawn across the preceding four. Finally we count the tally marks in each class and enter the figure at the appropriate place in the third column. This is shown in the following table.

TABLE 1·1

(1) Number of members in a family	(2) Tally Marks	(3) Frequency
2	𝍸 𝍸 |	11
3	𝍸 𝍸 |	11
4	𝍸 𝍸 |||	14
5	𝍸 𝍸 𝍸	15
6	𝍸 ||||	9
7	𝍸 𝍸	10
8	𝍸 |	6
9	𝍸 ||	7
10	||||	4
11	|	1
12	||	2
Total		90

The figures in column (3) (Frequency column) give the frequencies of the families with the corresponding numbers of members. Thus, the frequency of families with 4 members is 14; with 6 members is 9 with 7 members is 10; and so on. The last figure in column (3) gives the total frequency of the families observed.

(7·1) **Cumulative Frequencies.** The frequency distribution in Table 1·1 may be represented in a variety of other ways. For instance one may giver elative frequencies, i. e. the proportion of families with k members where $k = 2, 3, \ldots\ldots, 12$.

One may also ask with reference to the above distribution the following questions : " What is the number of families with 4 or more than 4 members ? Or what is the number of families with 5 or less than 5 members ? " To answer such questions we can form tables giving cumulative totals of frequencies beginning either from the lowest class upwards or from the highest class downwards. Cumulative totals, obtained by starting from the highest class downwards, are called cumulative frequencies (more than type); while cumulative totals,

obtained by beginning from the lowest class upwards, are called cumulative frequencies (less than type). We may thus give the following definitions.

Definitions. If $f_1, f_2, \ldots, f_n$ be the frequencies of the 1st, 2nd,..., nth class intervals, arranged in increasing order of magnitudes, of a frequency distribution, then

(i) $f_1, f_1 + f_2, \ldots, f_1 + \ldots + f_n$ are called the less than or equal to cumulative frequencies of the 1st, 2nd, ... , nth class intervals of the distribution and

(ii) $f_1 + f_2 + \ldots + f_n, f_2 + f_3 + \ldots + f_n, \ldots, f_{n-1} + f_n, f_n$ are called the greater than or equal to cumulative frequencies of the 1st, 2nd, ... , $(n-1)th$, nth class intervals of the distribution.

Reference may be made to " less than or equal to " or " more than or equal to " cumulative frequencies according to the requirements.

TABLE 1·2

Number of members	Frequency	Cumulative Frequency ($<$ or $=$, type)	Cumulative Frequency ($>$ or $=$, type)	Relative Frequency
2	11	11	90	0·12
3	11	22	79	0·12
4	14	36	68	0·16
5	15	51	54	0·17
6	9	60	39	0·10
7	10	70	30	0·11
8	6	76	20	0·07
9	7	83	14	0·08
10	4	87	7	0·04
11	1	88	3	0·01
12	2	90	2	0·02

For the data given in Table 1·1, we have the corresponding Table 1·2 giving frequencies, cumulative frequencies (less than or $=$, type) and cumulative frequencies (more than or $=$, type), and relative frequencies.

The proportions in the last column are correct to the second place of decimals and the sum of the figures in that column is 1.

We now consider another example to illustrate the formation of a frequency distribution of a discrete variable.

(8) Frequency Distribution for a Discrete Vairable (Grouped data).

The following are the marks obtained by 100 students in the subject of Algebra in a terminal examination, the marks being out of 100.

72, 81, 91, 73, 72, 96, 84, 67, 55, 85; 68, 75, 78, 81, 74, 84, 81, 72, 91, 90; 65, 64, 60, 62, 80, 85, 60, 91, 72, 82; 81, 71, 58, 64, 90, 85, 64, 78, 85, 85; 64, 84, 72, 60, 93, 60, 72, 78, 91, 81; 50, 80, 79, 83, 55, 55, 78, 84, 64, 73; 65, 67, 73, 74, 94, 72, 96, 78, 58, 69; 87, 69, 88, 67, 70, 67, 56, 67, 87, 56; 91, 72, 71, 64, 85, 87, 68, 72, 78, 56, 67, 74, 94, 90, 84, 84, 88, 84, 64, 50.

If we arrange these figures in order of magnitude we find that the discrete variable (viz. number of marks scored) assumes as many as 32 isolated discrete values. If we were to form a frequency distribution table in the above manner the work will be very laborious. We have, therefore, to group the data into convenient classes so that the data is summarised in a manageable form.

We observe that the domain of variation of the variable is the set of integers from 0 to 100. The highest number of marks scored is 96 and the lowest is 50. Thus the range of the data is $96 - 50 = 46$. In deciding on the grouping of the data into classes for reducing the data to a manageable form we observe that the number of classes should not be too large; for if that were so, the object of summarising would be defeated; nor should the number of classes be too small ,if that were so we will miss a good deal of detail available in the data and get a distorted picture. As a rule one should have between 10 and 25 classes, the actual number depending on the total frequeney. The classes should further be exhaustive, i.e., no observed value should be left out; they should also be non-overlapping so that no observed value falls in more than one class. It will be found usually convenient, though there are exceptions to the rule, to have all classes of the same length; i. e. each class can contain the same number of distinct values of the discrete variable.

Bearing these points in mind, we group the data into ten classes. Class one will contain marks from 47 to 51; class two will contain marks from 52 to 56; and so on. Finally class ten will contain marks from 92 to 96. These classes will be denoted as : 47–51; 52–56; ... ; 92–96.

We next proceed to construct a Table with three columns; the first column to represent the classes; the second to incorporate the tally-sheet and the third to give the corresponding frequency. Corresponding to each

class we draw a horizontal row. We go through the given data and corresponding to each value we put a tally mark against the class which includes the value read. Thus the tally mark corresponding to the value 60 is put against the class 57–61. Dealing with the data in this manner we get the Frequency Distribution Table 1·3.

TABLE 1·3

Frequency Distribution of Marks of 100 students

Class Interval	Tally sheet	Frequency
47–51	𝍸	2
52–56	𝍸 |	6
57–61	𝍸 |	6
62–66	𝍸 𝍸	10
67–71	𝍸 𝍸 |||	13
72–76	𝍸 𝍸 𝍸 |	16
77–81	𝍸 𝍸 ||||	14
82–86	𝍸 𝍸 𝍸	15
87–91	𝍸 𝍸 |||	13
92–96	𝍸	5
Total		100

Definitions

(1) Class-limits. *The values of the variate which separate the various classes are called class limits.*

The class-limits are the end points of the class intervals. Two types of class intervals are used. (i) Non-overlapping such as 47–51, 52–56... etc.

(ii) Overlapping such as 47–52, 52–57, ... etc.

Here the variate being discrete

47–52 means $47 \leqslant x < 52$, $52 \leqslant x < 57$, ... etc.

We have given in the Table 1·3 non-overlapping class intervals.

(2) **Class Marks.** *The mid-value of a class interval is called its class-mark.*

(3) **Length of a class interval.** *When the class intervals are all equal, the difference between two consecutive upper limits or lower limits or class marks is the width or the length of the class interval.*

We now illustrate these terms with reference to the table 1·3.

Symbols such as 47–51, or 52–56 etc. in the above frequency distribution are class **intervals.** In the case of the distribution of a discrete variable the length or the width of a class interval is by definition as many units as the number of distinct values of the discrete variable that the class can contain. Thus in the above frequency distribution the length of each class interval is 5 units and is the diffierence between the upper limits of two successive class-intervals.

For the class 47–51, the smaller number 47 is the **lower class limit** and the larger number 51 is the **upper class limit.**

The arithmetical average of the two class limits (viz the lower class limit and the upper class limit) is the central value or mid-value or the **class mark** of the class interval. In the above example the successive central values or the class marks of the successive class intervals are 49, 54,...,94. It may be remarked that for many purposes, like the calculation of the mean and variance which we shall consider later, it is convenient and customary to assume that the whole frequency in any class interval is concentrated at the central value (i. e. the class mark) of the class interval. Thus the class interval 67–71 has a frequency 13, which means that the number of observations falling in the class is 13; for many calculations we assume that all these 13 observations have the common value 69. This will, of course, not be accurate unless the 13 observations are spread over the class interval evenly. But it is often assumed that the positive and negative errors arising, when the spread of the observations is not uniform, would nearly cancel out, not vitiating seriously the results obtained.

In terms of class intervals and class marks, the above frequency distribution of the marks of 100 students may be exhibited in the form of the following table.

TABLE 1·4

Class-interval (Marks)	Class-Mark (x)	Frequency (f)
47–51	49	2
52–56	54	6
57–61	59	6
62–66	64	10
67–71	69	13
72–76	4	16
77–81	79	14
82–86	84	15
87–91	89	13
92–96	94	5
Total		100

(9) Frequency Distribution for a Continuous Variable.

(Grouped Data with equal Class Intervals)

We have seen how in the case of a discrete variable, taking only a few isolated values, one class could be taken for each different value of the variable for preparing a table of frequency distribution. When the discrete variable takes a fairly large number of different values we had to group the data into a number of classes to achieve the object of summarising the data in a manageable form.

In the case of a continuous variable, if we take one class for each different value of the variable, the number of classes would be unduly large, in fact infinite, since a continuous variable, by definition, assumes an infinite number of different possible values and the necessity of putting the data into a reasonable number of classes becomes imperative. The same general considerations, as in the case of a discrete variable, will apply, in this case also, as regards classification.

As an illustration, let us consider a frequency distribution given in Table 1 · 5 for the continuous variable represented by the weights, recorded to the nearest kilogram, of 100 workers working in a factory.

TABLE 1·5

Weights of 100 Factory Workers

Weight (in kg.)	Number of Workers
60–62	5
63–65	18
66–68	42
69–71	27
72–74	8
Total	100

From the observations recorded, a tally sheet is prepared and the information is summarized in the above table. Such data, as represented in Table 1·5, are often called grouped data.

(i) **Class Intervals and Class Limits :** A symbol defining a class such as 60–62 in the above table is called a **class interval**. The end numbers, 60 and 62, are called **class limits**; the smaller number 60 of the class 60–62 is called the **lower class limit** and the larger number 62 is called the **upper class limit.** The terms, class and class interval, are often used interchangeably although the class interval is actually a symbol for the class.

(ii) **Class Boundaries** : *For a cotinuous variate, if the class intervals are non-overlapping, then it is desirable to make them over-lapping. In general, these new over-lapping class-limits are called class-boundaries.* The weights are recorded to the nearest kilogram. The class interval 60-62 includes all measurements from 59·50000 to 62·499.. kg; the variable being a continuous one. These numbers, indicated briefly by the exact numbers 59·5 and 62·5 are called **class boundaries** or true **class limits**; the smaller number 59·5 is the **lower class boundary** and the larger number 62·5 is the **upper class boundary.**

In practice, the class boundaries are obtained by adding the upper limit of one class interval to the lower limit of the next higher class interval and dividing the sum by 2. Sometimes, class boundaries are used to symbolize classes. For example, the various classes in the first column of Table 1·5 could be indicated by 59·5–62·5, 62·5–65·5, etc. To avoid

ambiguity in using such notation, class boundaries should not coincide with actual observations. Thus if an observation were 62·5, it would not be possible to decide whether it belonged to the class interval 59·5–62·5 or 62·5–65·5. However, if such an occasion arises we would consider it as belonging to the latter class. Thus the Table 1·5 would be indicated thus :

Class Intervals	Observations
$59 \cdot 5 \leqslant x < 62 \cdot 5$	5
$62 \cdot 5 \leqslant x < 65 \cdot 5$	18
... ...	...
... ...	...

(iii) **The size or width of a Class Interval** : The size or width of a class interval is the difference between the lower and upper class boundaries and is also referred to as the **class width, class size or class length.**

If all class intervals of a frequency distribution have equal widths, this common width is equal to the difference between two successive lower class limits or two successive upper class limits.

In the case of Table 1·5 the class interval, say, c, is given by

$$c = 62 \cdot 5 - 59 \cdot 5 = 3.$$

(iv) **The Class Mark** : The class mark is the mid-value of the class interval and is obtained by adding the lower and upper class limits and dividing the sum by two.

For example, the class mark of the interval 60 – 62 is

$$\frac{60 + 62}{2} = 61.$$

If required, all observations belonging to a given class interval are assumed to coincide with the class mark. Thus all the weights in the class interval 60–62 are considered as 61 kg.

(v) **Cumulative Frequency** : The total frequency of all values less than or equal to the upper class boundary of a given class interval is called the **cumulative frequency** upto and including that class interval.

For example, the cumulative frequency up to and including the class interval 66– 68 in the above table is 5 + 18 + 42 = 65, thus signifying that 65 factory workers have weights less than 68·5 kg. For some purposes it is desirable to consider a cumulative frequency distribution of all values greater than or equal to the lower class boundary of each class interval. A Table representing such cumulative frequencies is often called cumulative frequency distribution less than type or greater than type as the case may be.

TABLE 1·6

Cumulative Frequency Distribution of Table 1·5

Class interval	Frequency	Cumulative Frequency Less than Type	Cumulative Frequency Greater than Type
59·5–62·5	5	5	100
62·5–65·5	18	23	95
65·5–68·5	42	65	77
68·5–71·5	27	92	35
71·5–74·5	8	100	8

Whenever we refer to cumulative distributions without particular reference then "the Less than Type" is implied.

Example : The following data gives the weights (to the nearest pound) of 40 students in a Higher Secondary Class. Construct a frequency distribution table giving frequencies and cumulative frequencies of the two types. Tabulate the data into equal classes, the class one being 118–122. Indicate also the class boundaries and the class marks.

144, 125, 149, 157, 132, 150, 164, 138, 146, 158,
140, 147, 168, 126, 138, 176, 136, 148, 152, 144,
163, 119, 154, 165, 146, 173, 161, 145, 135, 142,
142, 147, 135, 153, 140, 135, 150, 156, 128, 145.

The successive classes will be 118–122; 123–127; ..... ; 173–177. The lower and upper class boundaries for the class one will be 117·5 and 122·5 and the class-mark for this class will be

$$\frac{117\cdot5+122\cdot5}{2} \text{ or } \frac{118+122}{2} = 120.$$

The class boundaries and the class marks will go on increasing by 5.

TABLE 1·7

Tally-sheet for the weights of 40 students

Class	Tally marks	Total
118–122	|	1
123–127	||	2
128–132	||	2
133–137	||||	4
138–142	𝍸 |	6
143–147	𝍸 |||	8
148–152	𝍸	5
153–157	||||	4
158–162	||	2
163–167	|||	3
168–172	|	1
173–177	||	2
Total		40

We must prepare a tally-sheet in the usual way with 3 columns; the first column for the successive classes; in the second column we shall put the tally-strokes opposite to the classes to which they belong and the last column will give the totals of the tally-strokes for the various classes.

We now prepare the required frequency distribution table indicating the class boundaries and class marks in columns 1 and 2 respectively. Columns 3, 4, 5 will give the frequency, the cumulative frequencies of less than type and greater than type respectively.

TABLE 1·8

Class Boundaries	Class Mark	Frequency	Cumulative Frequency (less than type)	Cumulative Frequency (greater than type)
117·5–122·5	120	1	1	40
122·5–127·5	125	2	3	39
127·5–132·5	130	2	5	37
132·5–137·5	135	4	9	35
137·5–142·5	140	6	15	31
142·5–147·5	145	8	23	25
147·5–152·5	150	5	28	17
152·5–157·5	155	4	32	12
157·5–162·5	160	2	34	8
162·5–167·5	165	3	37	6
167·5–172·5	170	1	38	3
172·5–177·5	175	2	40	2
Total		40		

(10) Graphical Representation of a Frequency Distribution.

In order to convey statistical information to the general public, representation of statical data by diagrams, graphs, charts or pictures is more effective than tabular representation, it being easily intelligible to a layman. We will consider, at this stage, the graphical representation of a frequency distribution by Frequency bar chart, Frequency polygon, Histogram, Frequency curve and Cumulative frequency polygon (ogive). We will first give some definitions.

Definitions

(i) **Frequency Bar Chart.** We shall denote the discrete variable by x; the various values of the discrete variable will be denoted by $x_1, x_2, x_3, \ldots, x_n$ and their frequencies will be denoted by $f_1, f_2, f_3, \ldots, f_n$ respectively. To represent the data graphically, we take two rectangular co-ordinate axes. With a suitable scale the different values of the discrete variable x may be located by means of points on the x-axis. At each of these points, we draw a perpendicular to the x-axis to represent the corresponding frequency with a proper suitable scale. Such a diagram is called a **frequency bar chart.**

(ii) **Frequency Polygon**. With a suitable scale for the variate x along the x-axis and for the frequencies along the y-axis, the points $P_r(x_r, f_r)$ are located. The frequency polygon is obtained by joining the points $P_1(x_1, f_1)$, $P_2(x_2, f_2)$,....., $P_n(x_n, f_n)$ by line segments. In order to make the polygon closed it is conventional to take a value preceding the least value of the variate and a value following the largest value of of the variate with zero frequency. If these points are [illegible] and $P_{n+1}(x_{n+1}, 0)$, we get a closed frequency polygon $P_0P_1P_2P_3\ldots P_nP_{n+1}$ after joining these points, in order, by straight lines.

A frequency polygon can also be drawn for a grouped frequency distribution by considering the whole frequency of each class-interval as being concentrated at the class-mark (mid-value) of the class-interval. If $x_1, x_2, \ldots, x_n$ are the class-marks of the n classes with class frequencies $f_1, f_1, \ldots, f_n$ and x_0 and x_{n+1} denote the class marks preceding x_1 and following x_n taken with zero class frequencies, then the closed frequency polygon is obtained by joining these points $P_r(x_r, f_r)$, $r = 0, 1, 2, \ldots\ldots$, $(n+1)$, by straight lines.

(iii) **Histogram**. A histogram or frequency histogram is a diagram consisting of a set of rectangles having (i) bases on the x-axis with centres at the class-marks and lengths equal to the class-interval sizes, (ii) areas proportional to class frequencies.

If the class intervals have equal size, the heights of the rectangles are proportional to the class frequencies and it is then customary to take the heights numerically equal to the class frequencies. If class intervals do not have the same size, the heights of the rectangles must be found.

We have area of a rectangle = base × height.

Thus, the height of the rectangle $= \dfrac{\text{area of the rectangle}}{\text{length of the base}}$

Thus, the height of the rectangle

$$\propto \frac{\text{class-frequency}}{\text{length of the class-interval}}.$$

This gives the frequency per unit length of the class-interval and is known as the frequency density over that class-interval. Thus, the ordinate of every point on the top side of a histogram gives the frequency density for the corresponding value x of the variate.

(iv) **A Frequency Curve**. A free hand curve drawn through the successive vertices of a frequency polygon of a given frequency distribution is called the frequency curve of the given distribution.

(v) **An ogive or a cumulative frequency polygon.** A cumulative frequency polygon, known as ogive, is obtained by plotting the cumulative frequencies corresponding to the values of the variate to which they belong. In drawing an ogive (generally understood as referring to a cumulative distribution of less than type, if there is no other reference), the frequency is to be plotted at the upper-most limit of the class-interval and not at the mid-point as is done in the case of the frequency polygon of grouped data. Cumulative frequency polygon when smoothed out will be referred to as an ogive curve.

The histogram suffers from the defect that the diagram changes if the class intervals are changed. This defect, however, is not present in the frequency polygon. However, the frequency polygon also suffers from a defect since it shows somewhat sudden changes and a picture of uniform and regular variation in the magnitude is not obtained. This defect, however, is removed in frequency curves. An ogive runs more regularly than the ordinary frequency curve since it is not affected by the variations in the group intervals. Ogive will be found useful for finding the median and also the mode.

Illustrative Examples

Ex. 1. Construct (i) a histogram and (ii) a frequency polygon corresponding to the frequency distribution given in Table 1.5.

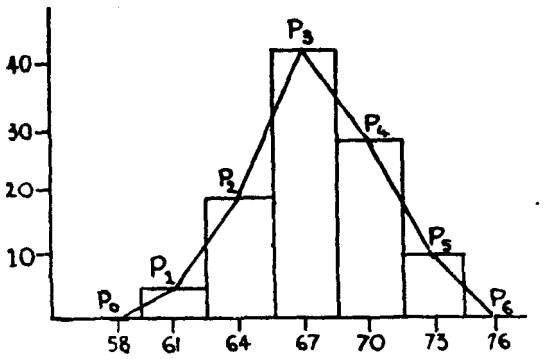


Fig. 12.1

The table shows the weight x, in kilograms, of 100 workers. The variate x, therefore, is a continuous one. We mark on the x-axis the class-marks 61, 64, 67, 70 and 73. The width of each class interval is three. Hence this data is a grouped data divided into classes of equal width. We now draw rectangles with equal bases on the x-axis having the width of each class-interval as a base with class-mark as the mid-point and the height proportional to the corresponding class frequency. Since the widths of all the class-intervals are equal, the heights of the rectangles can be taken to be the corresponding frequencies. The histogram is shown in Fig. 12.1. We plot the points P_1 (61, 5), P_2 (64, 18), P_3 (67, 42), P_4 (70, 27) and P_5 (73, 8). The abscissae

of the points are class-marks of the various class-intervals and the ordinates are equal to the corresponding class frequencies considered as centred at the class-marks. These points are joined by straight lines. It is customary to add the extensions P_0P_1 and P_5P_6 to the next lower and higher class-marks where at P_0 and P_6 the frequencies are each equal to zero. When the points $P_0, P_1, P_2, P_3, P_4, P_5, P_6$ are joined by line segments we get a closed frequency polygon $P_0P_1 \ldots P_6$. In this case, it may be noted that the sum of the areas of the rectangles in the histogram is equal to the total area bounded by the frequency polygon and the x-axis.

Ex. 2. Draw a 'equal to or less than type' ogive for data given in Table 1.6.

In order to draw a 'less than type' ogive we plot the points whose abscissae are the upper class boundaries of class-intervals and the corresponding ordinates are equal to the less than type cumulative frequencies of the corresponding class-intervals. Thus for the given data we plot the points (59.5, 0), (62.5, 5), (65.5, 23), (68.5, 65), (71.5, 92) and (74.5, 100). We join these points by a smooth curve in order to get the required ogive curve. Similarly, 'equal to or more than type' ogive can be drawn. (See Fig. 12.2).

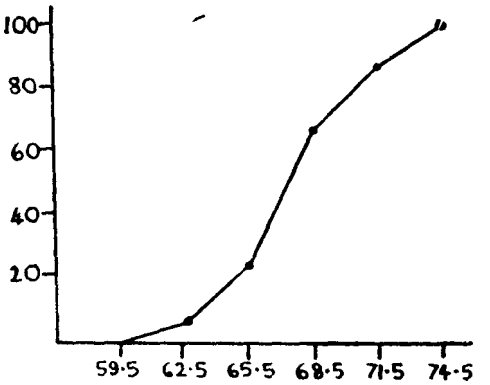


Fig. 12.2

Ex. 3. The following table shows a frequency distribution of the weekly wages, in Rs., of 65 employees in a factory.

TABLE 1.9

Wages (Rs.)	Number of Employees
50–59.99	8
60–69.99	10
70–79.99	16
80–89.99	14
90–99.99	10
100–109.99	5
110–119.99	2
Total	65

(a) With reference to this table,

determine (i) the lower limit of the sixth class;
(ii) the upper limit of the fourth class;
(iii) the class mark of the third class;
(iv) the class boundaries of the fifth class.

(b) Construct for this data
(i) a histogram, (ii) a frequency polygon.

(c) Construct for this data
(i) a cumulative frequency distribution;
(ii) an ogive (less than type);
(iii) 'equal to or more than' cumulative frequency distribution and the corresponding ogive.

(d) From the ogives, estimate the number of employees earning
(i) less than Rs. 88/-per week,
(ii) Rs. 96/- or more per week,
(iii) at least Rs. 63/-- but less than Rs. 75/- per week.

ANSWERS

(a) (i) Rs. 100. (ii) Rs. 89·99.
(iii) $\frac{1}{2}(70 + 79.99) = 74.995$, i. e. say 75.
(iv) Lower class boundary $= \frac{1}{2}(90 + 89.99) = 89.995$,
upper class boundary $= \frac{1}{2}(99.99 + 100) = 99.995$.

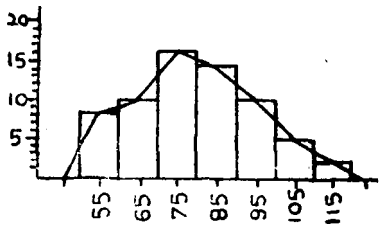


Fig. 12.3

(b) (i) The histogram is shown in Fig. 12.3.

(ii) The frequency polygon is indicated by the straight line graph in Fig. 12.3.

(c) (i) The cumulative frequency distribution (less than type) is shown in the following table.

TABLE 1.10

Wages (Rs.)	Cumulative Frequency (Less than type)
Less than 50	0
Less than 60	8
Less than 70	18
Less than 80	34
Less than 90	48
Less than 100	58
Less than 110	63
Less than 120	65

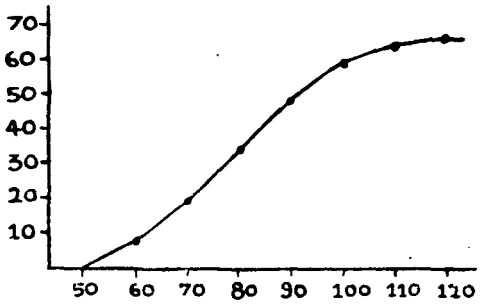


Fig. 12.4

(c) (ii). The ogive (less than type) is indicated in the Fig. 12.4.

(c) (iii). The cumulative frequency distribution (equal to or more than type) is given in the following table :—

TABLE 1.11

Wages (Rs.)	Cumulative Frequency (equa to or more than)
50 or more	65
60 or more	57
70 or more	47
80 or more	31
90 or more	17
100 or more	7
110 or more	2
120 or more	0

More than or equal to type ogive is shown in the following figure 12.5.

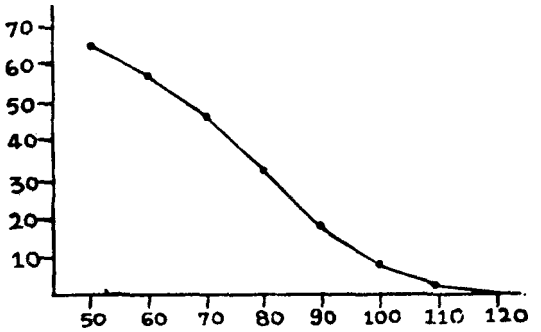


Fig. 12.5

(d) (i). Refering to the " less than " ogive in Fig. 12.4 we can construct a vertical line intersecting the "wages" axis at Rs. 88. This line will be seen to meet the ogive at the point with co-ordinates (88, 45). Hence 45 employees earn less than Rs. 88 per week.

(d) (ii). Referring to "equal to or more than" ogive in Fig. 12·5 we construct a vertical line at Rs. 96. This line will be seen to meet the ogive at the point (96, 11). Hencea 11 employees earn Rs. 96 or more. This can also be obtained from the 'less than type' ogive by constructing a vertical line at Rs. 96. From this we find that 54 employees earn less than Rs. 96. Hence 65 – 54 = 11 employees earn Rs. 96 or more.

(iii) Using the 'less than type' ogive of Fig. 12.4 we have

required number of employees

= (number earning less than Rs. 75) – (number earning less than Rs. 63)

= 26 – 11 = 15.

Ex. 4. Five coins were tossed 1000 times and at each toss the number of heads was observed. The number of tosses during which 0, 1, 2, 3, 4 and 5 heads were obtained is shown in the following table 1.12. Draw the bar chart of the frequency distribution.

TABLE 1.12

Number of heads	Number of tosses (Frequency)
0	38
1	144
2	342
3	287
4	164
5	25
Total	1000

We observe that the data consists of a discrete variate. Such data of a discrete variate can be shown by a bar chart, sometimes known as a rod graph. We take the points 0, 1, 2, 3, 4, 5 on the x-axis and erect, at each of these points, lines of zero width, equal in length to the corresponding frequencies. This is shown in Fig. 12.6

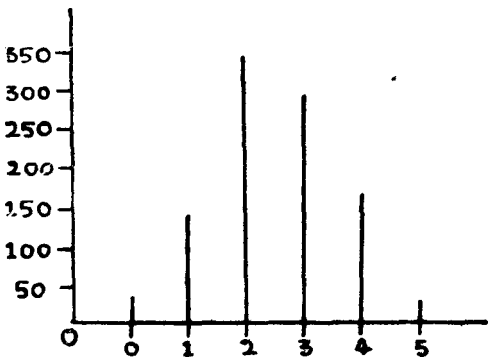


Fig. 12.6

EXERCISE 12

(1) The following data gives the marks in Statistics, obtained by forty students :

47, 32, 26, 43, 33, 21, 34, 28, 31, 33,
25, 43, 47, 35, 23, 27, 31, 41, 46, 33,
36, 32, 31, 27, 21, 38, 41, 22, 31, 39,
33, 37, 42, 23, 22, 28, 27, 31, 34, 30.

Prepare a frequency table, taking class-intervals 20-24, 25-29; etc. Draw a histogram for the distribution.

(2) The following data gives the number of people living in each of fifty buildings :

21, 50, 35, 39, 48, 46, 36, 54, 42, 30, 29, 42, 32, 40, 34,
31, 35, 37, 52, 44, 39, 45, 37, 33, 51, 53, 52, 46, 43, 47,
41, 26, 52, 48, 25, 34, 57, 55, 36, 27, 54, 36, 41, 33, 23,
39, 28, 44, 45, 38.

Prepare a frequency table taking class intervals 20-24; 25-29, 30-34; etc. Draw a a histogram for the distribution.

(3) The lengths, in centimeters, of 50 silver bars are as follows :

106, 82, 129, 204, 90, 107, 123, 139, 111, 107,
76, 125, 119, 141, 81, 82, 111, 115, 136, 131,
109, 92, 128, 123, 75, 107, 86, 101, 90, 84,
93, 126, 84, 98, 110, 187, 68, 99, 110, 81,
95, 130, 113, 78, 118, 115, 70, 186, 115, 104.

Form the grouped frequency table by dividing the variate range into intervals of equal width, each corresponding to 20 cms. in such a way that the mid-value of the first class corresponds to 70 cms.

(4) The following data gives the rents of 50 blocks. Prepare a frequency table taking class intervals 30-34, 35-39, 40-44, etc. Draw a histogram for the distribution :

38, 42, 49, 37, 82, 74, 75, 62, 54, 79, 84, 57, 36, 41, 47,
44, 63, 69, 45, 84, 47, 39, 48, 54, 61, 71, 72, 63, 80, 51,
31, 38, 43, 57, 67, 77, 74, 64, 34, 39, 71, 32, 39, 47, 53,
64, 73, 34, 39, 43.

(5) The weekly wages of 50 workers, in rupees, are given below. Prepare the frequency table with class intervals 25-29, 30-34,....., etc. Draw a frequency polygon to represent the data.

32, 41, 33, 37, 49, 53, 46, 67, 30, 27, 56, 51, 52, 47, 45,
36, 56, 49, 42, 44, 41, 45, 65, 38, 57, 62, 55, 48, 40, 64,
54, 52, 42, 56, 54, 40, 35, 46, 37, 41, 45, 50, 47, 52, 46,
48, 50, 58, 39, 46.

(6) The following statistics of height, in centimetres, were gathered in a certain class of 30 students.

131, 141, 149, 145, 136, 138, 133, 140, 147, 144,
137, 132, 130, 146, 145, 143, 142, 131, 136, 139,
138, 135, 134, 149, 146, 142, 143, 136, 140, 141.

For this data

(i) form a frequency distribution by items (i. e. without grouping):

(ii) draw a bar chart and the histogram;

(iii) plot the frequency polygon.

(7) Taking data given in example 6,

(i) form a frequency distribution with class intervals, 126-130, 131-135, 136-140 etc.

(ii) construct the histogram and frequency polygon.

(8) The weights in kilograms of 30 students in a class are given below. Form a frequency distribution by items (i. e. without grouping). Represent the data by a bar chart and a histogram drawing also the frequency polygon.

21, 30, 40, 25, 26, 22, 39, 31, 29, 36, 38, 35, 34, 33, 30, 23, 27, 27, 29
31, 33, 22, 21, 36, 40, 31, 33, 30, 37, 36.

(9) Taking the data given in example 8,

(i) form a frequency distribution with class intervals 20-24, 25-29, 30-34, ... etc.

(ii) Draw the frequency polygon and construct the histogram.

(10) The following table gives the distribution of 1000 families according to the number of children.

No. of children	0	1	2	3	4	5	6	7
No. of families	25	306	402	200	53	8	4	2

Draw the frequency polygon.

(11) Fill in the blanks in the following.

(i) If a character observed is not expressible in a numerical form but is qualitative, it is called an———.

(ii) A variate is expressible in a———form and is either——— or ———.

(iii) A tabular arrangement of data by classes with the corresponding class frequencies is called a——— ———.

(iv) The sum of the proportions of relative frequencies is———.

(v) The arithmetical mean of the two class limits of a class interval is called its ——— mark.

(vi) A cumulative frequency polygon is called an ——.

(vii) In a bar chart the length of the bar is proportional to the correspounding—

(viii) In drawing a histogram of a frequency distribution of a grouped data the areas of the rectangles are proportional to the class————.

(12) Complete the following statements by choosing the correct alternatives from " population, attribute, variate, sample, quantitative, interval, discrete. "

(i) The number of members of a family in a statistical investigation of a co-operative housing society is a —— variate.

(ii) A continuous variate is a——characteristic which is capable of taking each number in an——as its value.

(iii) A subset of a population which is actually investigated for convenience is known as a——.

(iv) A quantitative characteristic of observations to be studied in a statistical enquiry is called a——.

(v) A qualitative characteristic of the observations in the data is known as an ——.

(vi) The set of all objects to be investigated for a particular characteristic is called a——.

ANSWERS

(11) (i) Attribute. (ii) numerical, discrete, continuous. (iii) frequency distribution, (iv) one. (v) class, (vi) ogive. (vii) frequency, (viii) frequencies.

(12) (i) Discrete, (ii) quantitative, interval, (iii) sample, (iv) variate, (v) attribute, (vi) population.

□ □ □

CHAPTER 13

Measures of Central Tendency

(1) Measures of Central Tendency.

Average. Definition. *A single value which can be considered as typcial or representative of a set of observations and around which the observations can be considered as centred is called an average (or an average value) or a centre of location.*

Since the typical values tend to lie centrally within a set of observations when arranged according to magnitude, averages are called measures of central tendency. The fact that distributions have a peak (average) value, about which the observations are more or less symmetrically distributed, is of great importance both theoretically and practically. We, therefore, consider in this chapter the methods of specifying the average value and propose to discuss later the methods of determining how the individual observations are dispersed or spread or scattered about the average value in a set of observations.

The fundamental measures of central tendency are (i) Arithmetic Mean, (ii) Median, (iii) Mode, (iv) Geometric Mean, (v) Harmonic Mean, (vi) Weighted Averages. However, the most common measure of central tendency or location is the Arithmetic Mean. We, therefore, propose to consider first the Arithmetic Mean.

(2) Arithmetic Mean. The Arithmetic mean or average is the value obtained by the simple process of adding all the observations together and dividing the sum by the number of observations.

Let us first consider ungrouped data. Suppose we have n observations $x_1, x_2, x_3, \ldots, x_n$; then the Arithmetic Mean obviously is

$$\frac{x_1 + x_2 + \ldots\ldots + x_n}{n}.$$

We shall use the symbol $\bar{x}$ (pronounced as x bar) to denote the Arithmetic Mean. Since we have to write very frequently the sum of observations, we use the usual symbol 'Σ' (pronounced as sigma) to denote the sum. The symbol x_i will be used to denote, in general, the i th

observation. The sum of n observations viz. $x_1 + x_2 + \ldots\ldots + x_n$ will be represented symbolically by $\sum_{i=1}^{n} x_i$.

Definition. The Arithmetic mean, denoted by $\bar{x}$, of a set of observations $x_1, x_2, \ldots\ldots, x_n$ is defined by

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} . \qquad \ldots \quad (i)$$

For example, the Arithmetic Mean (sometimes called only the Mean) of observations 3, 7, 7, 5, 4 is given by

$$\bar{x} = \frac{3+7+7+5+4}{5} , \text{ i. e. } \frac{26}{5} \text{ or } 5{\cdot}2.$$

There is, of course, no difficulty in finding the mean of a small number of observations. However, if the number of observations is very large, the arithmetic for calculating the mean becomes laborious. Various short-cut methods are used to lighten the labour of calculations.

One of these methods is to group the observations in a frequency distribution.

Definition. *If the observations $x_1, x_2, \ldots\ldots, x_n$ are repeated $f_1, f_2, \ldots\ldots, f_n$ times respectilvey then we define the Arithmetic Mean, denoted by, $\bar{x}$ by*

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum f_i} , \qquad \text{i. e. } \frac{f_1 x_1 + f_2 x_2 + \ldots\ldots + f_n x_n}{f_1 + f_2 + \ldots\ldots + f_n} . \qquad \ldots \quad (ii)$$

The symbol $\sum_{i=1}^{n}$ may sometimes be written simply as Σ, the range over which the addition is to be performed being understood from the context.

We may illustrate the process by the following example:

Illustration :-- Find the mean of the following 50 observations.

19, 19, 20, 20, 20, 19, 21, 19, 20, 18, 20, 20, 19, 19, 20,
19, 21, 19, 19, 21, 18, 20, 18, 18, 17, 20, 20, 22, 20, 20,
20, 20, 20, 21, 20, 17, 23, 18, 17, 21, 20, 21, 20, 20, 21,
18, 21, 19, 20, 19.

We may tabulate the given observations as given in the following ungrouped frequency table.

TABLE 2.1

Observation (x)	Frequency (f)	fx
17	3	51
18	6	108
19	11	209
20	20	400
21	8	168
22	1	22
23	1	23
Total	50	981

The first column shows the value of the observation. The second column shows the frequency with which each observation is repeated. The third column gives the product of the observed value and its frequency.

The Arithmetic Mean,

$$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{981}{50}, \text{ i. e. } \frac{1962}{100}, \text{ i. e. } 19.62.$$

(3) Calculation of the Mean by Choosing a Convenient Origin.

Another device which is sometimes useful is to subtract a constant from each of the observations. Each observation is thus measured from some convenient value. This convenient value may be taken as the value of the observation having the greatest frequency. Suppose in a set of observations the obseration having a value a has the greatest frequency. We then write the values of the observations as

$$x_1 - a, \; x_2 - a, \; \ldots\ldots, \; x_n - a.$$

Let, in general, $u = x - a$.

We have $\sum_{i=1}^{n} u = \sum_{i=1}^{n} (x_i - a)$

$$= (x_1 - a) + (x_2 - a) + (x_3 - a) + \ldots + (x_n - a)$$

$$= (x_1 + x_2 + \ldots + x_n) - na.$$

$$\therefore \quad \bar{u} = \frac{1}{n} \sum_{1}^{n} u_i = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} - a = \bar{x} - a.$$

Hence, we have the formula, $\bar{x} = a + \bar{u}$, where a is some convenient origin and

$$\bar{u} = \frac{1}{n} \Sigma u_i, \text{ where } u_i = x_i - a.$$

We may illustrate this process by solving the illustrative example of the last article again by this process. We may prepare the table as 2·2.

TABLE 2·2

x	$u = x - a$ ($a = 20$)	f	fu
17	−3	3	− 9
18	−2	6	−12
19	−1	11	−11
→20	0	20	0
21	1	8	8
22	2	1	2
23	3	1	3
Total		50	−19

We have from the Table 2·2,

$$\bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i} = \frac{-19}{50} = -\frac{38}{100} \text{ i. e. } -\cdot 38.$$

Hence $\bar{x} = a + \bar{u} = 20 - \cdot 38 = 19 \cdot 62$, as before.

(4) Calculation of Arithmetic mean from grouped data with equal class-intervals.

Change of origin and scale. (Coding method).

Suppose that all class intervals are of the same width, say, c. We measure all the class marks from some convenient value, say a, which generally should be taken as the class mark of a class having the maximum frequency or of a class which is the middle one. The difference of any two class marks happens to be a multiple of c, since all the class intervals are equal. We consider class frequencies as being centred at the corresponding class marks. With these assumptions we now prove the following :

Theorem : If $x_1, x_2, \ldots\ldots, x_n$ are n values of the class marks with frequencies $f_1, f_2, \ldots\ldots, f_n$ respectively and if each x_i is expressed in terms of the new variable u_i by the relation $x_i = a + cu_i$, then, with the usual notation, we have $\bar{x} = a + c\bar{u}$.

Proof :— We have, by definition,

$$\bar{x} = \frac{\Sigma f_i x_i}{N}, \text{ where } N = \Sigma f_i.$$

$$\therefore \quad \bar{x} = \frac{\Sigma f_i (a + cu_i)}{N}, \text{ since } x_i = a + cu_i,$$

$$= \frac{a \Sigma f_i + c \Sigma f_i u_i}{N}, \text{ since } a \text{ and } c \text{ are constants}$$

$$= \frac{aN + cN\bar{u}}{N}, \text{ since } \Sigma f_i = N \text{ and } \frac{\Sigma f_i u_i}{N} = \bar{u}.$$

$$\therefore \quad \bar{x} = a + c\bar{u}.$$

This method of obtaining the mean is called the **Coding method** for computing the mean. It is a very short method and should always be used for grouped data where class intervals are equal. Note that in the Coding method the values of the variate x (class marks) are transformed by changing the origin and scale into the values of the variate, say, u by the formula $x = a + cu$, where c is necessarily positive.

We illustrate the use of this formula by the following examples.

TABLE 2·3
Frequency distribution of marks of 100 students

Marks	Frequency
0–9	5
10–19	7
20–29	12
30–39	15
40–49	19
50–59	22
60–69	10
70–79	6
80–89	4
Total	100

Illustration 1. Find the Arithmetic mean of the marks in Mathematics, given in the Table 2.3, obtained by 100 students in a certain examination.

We prepare the following table :—

TABLE 2·4

Class mark x_i	Frequency f_i	$u_i = \left(\frac{x_i - a}{c}\right)$ $(a = 44{\cdot}5,\ c = 10)$	$f_i u_i$
4·5	5	−4	−20
14·5	7	−3	−21
24·5	12	−2	−24
34·5	15	−1	−15
→44·5	19	0	0
54·5	22	1	22
64·5	10	2	20
74·5	6	3	18
84·5	4	4	16
Total	100		−80+76 = −4

The first column gives the class marks. Each class mark is written by writing the mean of the lower and upper class limits of the corresponding class. Note that we have selected the value of a as 44.5 since this class is the middle one. Generally, the class mark which has the greatest frequency or which is the middle one should be taken as the value of a. Note that the class width is 10 and is equal for all the classes and is here, the value of c.

The entries in column 3 are obtained as :

$$u_1 = \frac{x_1 - a}{c} = \frac{4{\cdot}5 - 44{\cdot}5}{10} = \frac{-40}{10} = -4.$$

$$u_2 = \frac{x_2 - a}{c} = \frac{14{\cdot}5 - 44{\cdot}5}{10} = \frac{-30}{10} = -3,$$

... ...etc.

We now first find $\bar{u}$.

We have $\bar{u} = \frac{\Sigma f_i u_i}{N} = \frac{-4}{100} = -0{\cdot}04.$

Now we have the formula, $\bar{x} = a + c\,\bar{u}$;

$\therefore\ \bar{x} = 44.5 + 10(-0{\cdot}04) = 44{\cdot}5 - .4 = 44.1.$

Illustration 2. Find the mean weight in kilograms of 100 workers from the data given in the Table 1.5 of chapter 12.

We prepare the following table.

TABLE 2.5

Class marks x_i	Frequency i	$u_i = \left(\frac{x_i - a}{c}\right)$ $(a = 67,\ c = 3)$	$f_i u_i$
61	5	−2	−10
64	18	−1	−18
67	42	0	0
70	27	1	27
73	8	2	16
Total	100		−28 + 43 = 15

The first column gives the class mark.

First class mark $= \frac{60 + 62}{2} = \frac{122}{2} = 61,$

2nd class mark $= \frac{63 + 65}{2} = \frac{128}{2} = 64;$ ... ... etc.

There are five classes and the third class is the middle one and accidentally it also has the maximum frequency, namely, 42. We, therefore, choose $a = 67$. The class width is $64 - 61 = 3$ and is the same throughout.

We find $u_1 = \frac{x_1 - a}{c} = \frac{61 - 67}{3} = \frac{-6}{3} = -2$,

$u_2 = \frac{x_2 - a}{c} = \frac{64 - 67}{3} = -1$, ... etc.

This gives us the third column. The fourth column is obtained by multiplying the corresponding elements in the 2nd and 3rd columns

We find, $\bar{u} = \frac{\Sigma f_i u_i}{N} = \frac{15}{100} = .15$.

We have from the formula, $\bar{x} = a + c\bar{u}$,

$\bar{x} = 67 + 3(.15) = 67 + .45 = 67.45$.

(5) Weighted Arithmetic Mean

Definition. *If we associate with the numbers $x_1, x_2, \ldots, x_n$ certain weighting factors or weights, say, $w_1, w_2, \ldots, w_n$ respectively depending on the significance or importance attached to the numbers, then the weighted arithmetic mean $\bar{x}$ is defined by*

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \ldots + w_nx_n}{w_1 + w_2 + \ldots w_n}, \text{ i. e. } \frac{\Sigma w_i x_i}{\Sigma w_i}.$$

Illustration : Suppose that an owner of a workshop employs 6 men, 4 women and 2 boys. Considering the output, it is found that a woman or a boy cannot be reckoned as equal to a man. The owner, therefore, must decide what 'weight' must be given to each in proportion to a man. If a woman's work be taken, for example, to be three-quarters as effective and a boy's work to be half as effective as that of a man, we have the appropriate proportional weights as

$$1 : \frac{3}{4} : \frac{1}{2} \text{ or } 4 : 3 : 2.$$

Thus 6 men, 4 women and 2 boys would, on the average, be equivalent, in output, to

$$\left(6 + 4.\frac{3}{4} + 2.\frac{1}{2}\right) \text{ men} = 10 \text{ men}.$$

Here, $1, \frac{3}{4}, \frac{1}{2}$ are the weights because they tell us what weight should be given to each separate worker in calculating the average.

Example : If a preliminary examination is weighted three times as much as a test examination and a student obtains 85 marks in the preliminary examination and 70 and 90 marks in the test examinations in a subject then the mean marks are

$$= \frac{1.70 + 1.90 + 3.85}{1 + 1 + 3} = \frac{415}{5} = 83.$$

(6) Merits and Demerits of Arithmetic Mean

As a measure of location or central tendency, let us consider the advantages and disadvantages of the Arithmetic mean. Firstly, the mean is given by a specific formula and hence is easy to calculate. Some observations are bound to be greater and some less than the mean and hence it has a central place among the observations. Further, if all the observations are increased or decreased by the same number the mean is also increased or decreased by the same number. From these points of view this mean is a very good measure of average but it has some defects also. For example, a single observation widely different from other observations affects the mean considerably especially when the number of observations is very small. Consider a broker who earns the brokerage of Rs. 150, Rs. 200, Rs. 170, Rs. 180, Rs. 190, Rs. 170 and Rs. 2,503 in a week. The arithmetic mean of his daily earnings would be

$$\frac{1}{7}\left[\ 150 + 200 + 170 + 180 + 190 + 170 + 2503\ \right],$$

i. e. $\frac{3563}{7}$, i. e. 509.

Evidently Rs. 509 is a misleading figure for his daily earnings. The brokerage of Rs. 2503 on one day has affected the mean considerably and this mean cannot be taken as the daily average income of the broker. In certain data, there may not be any observation which is equal to the mean. Consider three married couples having 1, 3 and 4 children. The mean in this case will be

$$\frac{1 + 3 + 4}{3}, \text{ i. e. } \frac{8}{3},$$

and this is not equal to any observation. We have, therefore, to consider some other measures of location which may be free from these defects.

(7) The Median.

(7·1) The Median for ungrouped data. Definition. *The median of a set of observations is the middle value (or the arithmetic mean of the two middle values when the observations are even in number) when they are arranged in ascending or descending order of magnitude.*

Rule 1. If the number of observations, say n, is odd, then $\left(\left[\frac{n}{2} \right] + 1 \right)$ th observation, where $\left[\frac{n}{2} \right]$ denotes the greatest integer contained in $n/2$, is the middle observation and hence is the median,

the observations being arranged in order of magnitude. In this case $\left(\frac{n+1}{2}\right)$th observation becomes the median.

Rule 2. If n is even, there is no single middle observation.

The median, in this case, is the Arithmetic mean of $\frac{n}{2}$ th and $\left(\frac{n}{2}+1\right)$ th observations when the observations are arranged in order of magnitude.

Illustrations

Ex. (i). Consider the set of numbers 3, 4, 4, 5, 6, 8, 8, 8, 10. There are 9 observations. The number of observations is odd. By the first Rule or by inspection the 5th observation is the middle one. We note that the observations are already arranged in ascending order of magnitude. Hence the median is 6.

Ex. (ii). The marks in 6 subjects of a student are 84, 91, 72, 68, 87 and 78. Find the median of the marks.

We observe that the marks are not arranged in order of magnitude. We arrange them, say, in ascending order of magnitude : 68, 72, 78, 84, 87, 91. Since the number of observations is 6 and is even, by the 2nd Rule or by inspection the median is $\frac{1}{2}(78 + 84)$, i. e. 81.

(7·2) The Median for grouped data. For grouped data the median is obtained by interpolation and is given by the following formula :

$$\text{Median} = l_1 + \left[\frac{\frac{n}{2} - (\Sigma f)_{l_1}}{f_m}\right] c$$

where l_1 = lower class boundary of the median class,

n = total frequency, i. e. total number of items in the data,

$(\Sigma f)_{l_1}$ = sum of frequencies of all classes lower than the median class,

f_m = frequency of the median class,

c = size of the median class interval.

Geometrically the median is the value of x (abscissa) corresponding to that vertical line which divides the histogram into two parts having equal areas.

Illustration. Find the median of the distribution of weights of 40 students given in the following table.

Wt. (in lbs.)	118 to 126	127 to 135	136 to 144	145 to 153	154 to 162	163 to 171	172 to 180	Total
Frequency	3	5	9	12	5	4	2	40

The median is that weight for which half the total frequency, i. e. 20, lies above it and half lies below it. Since the sum of the first three and the first four class frequencies are respectively 17 and 29, therefore, the median class is the fourth, i. e. 145 to 153.

Now l_1 = lower class boundary of the median class = 144·5, the variate of the data being continuous.

n = the total frequency = 40.

$(\Sigma f)_{l_1}$ = sum of the frequencies of all classes lower than the median class = 17.

f_m = frequency of the median class = 12.

c = size of the median class interval = 9.

$\therefore$ by using the formula,

$$\text{Median} = 144.5 + \left[\frac{\frac{40}{2} - 17}{12}\right] \times 9 = 144.5 + \frac{3}{12} \times 9$$

$$= 146.8 \text{ lbs.}$$

Remark. Median is sometimes denoted by writing the symbol $\sim$ on the top of x as $\tilde{x}$ (read as x-curl).

(8) Merits and Demerits of the Median. We do not have to do any calculations while determining the median of a set of observations. However, we may require more time to determine the median than the mean since the process of ordering is time consuming if there are many observations. The median is not affected by a single abnormal observation as the mean and can be found very easily and quickly especially when the number of observations is very small. It is sufficient to arrange in magnitude only half the number of the observations to determine the median.

(9) Mode. Definition. *The mode of a set of observations is the value of that observation which occurs with the greatest frequency.*

Illustrations.

(i) The mode of the set of observations 2, 2, 5, 7, 9, 9, 9, 10, 10, 11, 12, 18 is 9 since the observation 9 alone is repeated thrice and no other observation is repeated more than thrice.

(ii) The set of observations 3, 5, 9, 11, 13, 17, 21 has no mode.

(iii) The set of observations 2, 3, 4, 4, 4, 5, 6, 7, 7, 7, 8 has two modes.

One can easily see that the mode of a set can be easily determined when it exists and then it would generally represent the average. There is no mode if no observation is repeated. However, if two or more observations are repeated the same number of times then the modal value cannot be uniquely determined.

We do not consider, here, a formula for determining the mode for a grouped data. If a frequency curve is constructed to fit the data, the mode will be the value (or values) of x (abscissa) corresponding to the point (points) having the maximum ordinate (ordinates). However, we give the following empirical formula for calculating the mode which can be conveniently used when the mean and median are known.

$$\text{Mode} = 3\,(\text{Median}) - 2\,(\text{Mean}).$$

(10) We may just mention some few more averages.

(i) Geometric Mean. If $x_1, x_2, x_3, \ldots, x_n$ are n observations, then their geometric mean is defined as

$$\sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdots x_n}, \quad x_i > 0, \; i = 1, 2, \ldots\ldots, n.$$

(ii) Harmonic Mean. If $x_1, x_2, \cdots, x_n$ are n observations, then their harmonic mean H is defined by

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}\right], \; x_i \neq 0.$$

(iii) Root Mean Square or Quadratic Mean. If $x_1, x_2, \ldots, x_n$ are n observations, then the quadratic mean is defined as

$$\frac{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}{n}$$

Even though various averages have been defined we will mainly use, in this book, the Arithmetic mean.

(11). Properties of the Arithmetic Mean.

Property (1). The algebraic sum of the deviations of a set of numbers from their arithmetic mean is zero.

Let $x_1, x_2, \ldots\ldots, x_n$ be a set of n numbers with $\bar{x}$ as their arithmetic mean.

The deviations of the numbers from their arithmetic mean are

$$x_1 - \bar{x}, \quad x_2 - \bar{x}, \ldots\ldots, \quad x_n - \bar{x}.$$

We use the symbol d_i to denote the deviation of the i th number x from the arithmetic mean $\bar{x}$ so that $d_i = x_i - \bar{x}$, for $i = 1, 2, \ldots, n$.

We have $\sum_{i=1}^{n} d_i = \sum_{1}^{n} (x_i - \bar{x})$

$$= (x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots + (x_n - \bar{x})$$
$$= (x_1 + x_2 + \cdots + x_n) - n\bar{x}$$
$$= n\bar{x} - n\bar{x}, \text{ since } \Sigma x_i = n\bar{x}$$
$$= 0.$$

Property (2). If the number of observations in two groups be n_1 and n_2 with means m_1 and m_2 respectively and if the two groups are pooled together, then the mean m of the $(n_1 + n_2)$ observations is given by

$$m = \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2}.$$

Let the n_1 observations add up to s_1 and the n_2 observations add up to s_2.

By the definition o the arithmetic mean, we have

$$m_1 = \frac{s_1}{n_1} \text{ and } m_2 = \frac{s_2}{n_2}.$$

$$\therefore s_1 = n_1 m_1 \text{ and } s_2 = n_2 m_2.$$

Now all the $(n_1 + n_2)$ observations add up to $s_1 + s_2$. Hence the arithmetic mean of the pooled group, say m, is given by

$$m = \frac{s_1 + s_2}{n_1 + n_2} = \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2}.$$

This result can be generalized as follows : If there are k groups with observations $n_1, n_2, \ldots, n_k$ with group means $m_1, m_2, \ldots, m_k$ respectively and if all the groups are pooled together then the mean of the pooled group is

$$\frac{n_1 m_1 + n_2 m_2 + \ldots\ldots + n_k m_k}{n_1 + n_2 + \ldots\ldots + n_k}.$$

Property (3). If a is any guessed or assumed arithmetic mean (which may be any number) and if $d_i = x_i - a$ be the deviation of x_i from a, then the formulae for the arithmetic mean

$$\text{(i) } \bar{x} = \frac{\Sigma x_i}{n}, \text{ (ii) } \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}.$$

can be, respectively, written as

$$\text{(i) } \bar{x} = a + \frac{\Sigma d_i}{n}, \text{ (ii) } \bar{x} = a + \frac{\Sigma f_i d_i}{\Sigma f_i}.$$

Note that both the formulae can be given by the equation

$\bar{x} = a + \bar{d}$, with the usual notation.

(i) We have $d_i = x_i - a$, $\therefore x_i = a + d_i \ldots$ (i)

Now $\bar{x} = \frac{\Sigma x_i}{n}$, where n denotes the total number of observations and Σ is used for $\sum_{i=1}^{n}$ for brevity.

$$\therefore \bar{x} = \frac{\Sigma (a + d_i)}{n} = \frac{\Sigma a + \Sigma d_i}{n}$$

$$= \frac{na + \Sigma d_i}{n} = a + \frac{\Sigma d_i}{n} = a + \bar{d}.$$

(ii) $$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{\Sigma f_i (a + d_i)}{\Sigma f_i} = \frac{a \Sigma f_i + \Sigma f_i d_i}{\Sigma f_i}$$

$$= a + \frac{\Sigma f_i d_i}{\Sigma f_i} = a + \bar{d}.$$

Illustrative Examples

Ex. 1. The mean annual salary paid to all employees in a factory is Rs. 30,000/-. The mean annual salary of male employees is Rs. 40,000/- whereas the mean annual salary of female employees is Rs. 10,000/- Find the ratio of the male and female employees.

Let n_1 and n_2 denote the numbers of male and female employees.

Let m_1 and m_2 denote the mean annual salaries of male and female employees.

Let m denote the mean annual salary of all the employees.

We have $m = \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2}$, $\therefore$ $30,000 = \frac{n_1 (40,000) + n_2 (10,000)}{n_1 + n_2}$.

$\therefore$ $3 (n_1 + n_2) = 4n_1 + n_2$. $\therefore$ $2n_2 = n_1$, i. e. $n_1 : n_2 = 2 : 1$.

Ex. 2. The average weight of 50 parcels was found to be 95 kg. It was found afterwards that one weight was wrongly recorded as 89 kg. instead of 98 kg. Find the correct average weight.

Total weight of 50 parcels from the given average $= 50 \times 95 = 4750$ kg.

The weight of one parcel was taken less by $(98 - 89)$, i. e., 9 kg.

$\therefore$ the correct total weight of 50 parcels $= 4750 + 9 = 4759$ kg.

$\therefore$ the correct average weight $= (4759) \div (50) = 95.18$ kg.

EXERCISE 13

(1) Four groups of students, consisting of 15, 20, 10 and 17 individuals reported mean weights of 162, 148, 153 and 140 pounds respectively. Find the mean weight of all the students.

(2) The mean m_1 of a group of 10 observations x_i $(i = 1, 2, \ldots, 10)$ is 9. Two new observations $x_{11} = 13$, $x_{12} = 17$ are added to the group. Find the mean of the 12 observations.

(3) The average height for a group of 40 persons was calculated to be 165 cms. It was later found that one of the observations was wrongly recorded as 154 cms., the correct value of which was 164 cms. Find the correct average height for the group in the light of the above correction.

(4) The following data represents the weight in lbs. of 100 persons. Find the mean weight.

Class limits (lbs)	80–89	90–99	100–109	110–119	120–129	130–139	140–149
Frequency	7	15	21	25	15	12	5

(5) Compute the mean weekly wage of the 65 employees from the data given in the illustrative Ex. 3, table 1.9 in Chapter 12.

(6) The hourly remunerations of five employees in an office are Rs. 2.52, Rs. 3.96, Rs. 3.28, Rs. 9.20 and Rs. 3.75. Find (i) the median hourly remuneration. (ii) the mean hourly remuneration.
State with reasons whether the median or the mean is a better average for the given data.

(7) Find the mean, median and mode for the set of numbers
3, 5, 2, 6, 5, 9, 5, 2, 8, 6.

(8) The mean of the ages of n boys is M and the mean of the ages of n' girls is M'. Find the mean of the ages of the boys and girls combined.

The mean annual salary paid to all employees in a company was Rs. 50,000. The mean annual salaries paid to male and female employees of the company were Rs. 52,000 and Rs. 42,000 respectively. Find the percentages of males and females employed by the company.

(9) Find the median for the following frequency distribution :

x	f	less than cumulative frequency
1	6	6
2	2	8
3	11	19
4	16	35
5	8	43
6	5	48

[When the data is given in an ungrouped frequency table and the size of the data is large it is not advisable to write all the observations in order of magnitude. In such cases the following process may be adopted.

Here $n = 48$, an even number. $\therefore$ Median $= \frac{1}{2}\left(x_{24} + x_{25}\right)$.

We want to find the 24th and 25th values in order of magnitude. From the table we find that 19 observations are less than or equal to $x = 3$ and 35 observations are less than or equal to $x = 4$. We, therefore, conclude that each of $x_{20}, x_{21}, x_{22}, \ldots\ldots, x_{35}$ has a value 4.

$\therefore$ required median $= \frac{1}{2}(x_{24} + x_{25}) = \frac{4+4}{2} = 4$.

(10) Find the median for the following distribution

Variable 'x'	5	7	9	11	13	15	17	19
Frequency 'f'	1	2	8	9	11	8	5	4

(11) Find the median for the frequency distribution of weekly wages, in Rs., of 65 employees given in Table 1.9 in Chapter 12.

(12) Find the median for the frequency distribution of weights of 100 factory workers given in Table 1.5 of Chapter 12.

(13) The arithmetic mean of n values of a variate is m. The sum of the first $(n-1)$ variates is s. Show that the value of the nth variate is $nm - s$.

(14) The sum of the deviations of n observed values of a variate from an arbitrary mean a is d. Show that the arithmetic mean of the observations is

$$\left(a + \frac{d}{n}\right).$$

(15) Fill in the blanks in the following

(i) The mean of 9 scores is 10 and of 6 other scores is 5.

$\therefore$ the mean of the 15 scores is.....

(ii) The mean of a set of 8 observations is 8. Two new observations 10 and 16 are added to the set.

$\therefore$ the mean of the 10 observations is .....

(iii) The sum of the deviations of 10 observations from 7 is 20.

$\therefore$ the mean of the distribution is.....

(iv) The mean age of the students in a class is 18 years. The mean age of boy students is 18.4 years and of girl students is 17.4 years.

$\therefore$ the percentage of boy students is .....

(v) The monthly earnings of 7 workers are given in rupees as 250, 290, 338, 250, 922, 340, 250.

For this data the mean, median and mode are respectively ....., ..., ... and ..... is the best average for this data.

(16) Choose the correct answer from the given alternatives in the following.

(i) Each of n boys possesses $5n$ rupees and each of $7n$ girls has $7n$ rupees. Hence the mean amount in rupees possessed by them is

(a) n, (b) $12n$, (c) $8.125n$, (d) none of (a), (b) and (c).

(ii) The mean of 10 observations is 7 and the mean of 20 other observations is 7·5. Hence the mean of all the observations is

(a) $\frac{7 + 7{\cdot}5}{2}$, (b) $\frac{10(7) + 20(7{\cdot}5)}{30}$, (c) $\frac{7(20) + 7{\cdot}5(10)}{30}$.

(iii) The mean of 5 scores is 20. If 3 is subtracted from each score, then the new mean is

(a) 35, (b) 23, (c) 17, (d) 5.

(iv) The median of 20 observations is 66. If 10 are added to each observation then the new median is

(a) 56, (b) 66, (c) 76, (d) none of (a), (b) and (c)

(v) Three students out of 9 secured more than 70 marks and two got less than 45 marks. If the remaining 4 students secured 50, 55, 60, 65 marks, then the median of their marks is

(a) 45, (b) 70, (c) 60, (d) none of (a), (b) and (c).

ANSWERS

(1) 150 pounds. **(2)** 10. **(3)** 165.25 cms. **(4)** 112.7 lbs. **(5)** Rs. 79.77.
(6) Median : Rs. 3·75; mean : Rs. 4.54. The median in this case gives the better average since it is not affected by the extreme observation of Rs 9.20.
(7) Mean : 5.1, median : 5, mode : 5.

(8) $\frac{nM + n'M'}{n + n'}$; males 80% and females 20%.

(10) 13. **(11)** Rs. 79.1. (12) 67.43 kilograms.

(15) (i) 8, (ii) 9, (iii) 9, (iv) 60, (v) 377.14, 290, 250; mode viz. Rs. 250.

(16) (i) (c), (ii) (b), (iii) (c), (iv) (c), (v) (c).

□

CHAPTER 14

Measures of Dispersion

(1) Dispersion. Definition. The degree to which numerical data tend to spread about an average value is called the variation or dispersion of the data.

Various measures of dispersion or variation are provided, the most common being the range, the mean deviation and the standard deviation.

In order to give a proper idea about the overall nature of the given set of observations, it is necessary, besides mentioning the average value, to state how scattered the given va ues are about the average.

Consider the recorded data of the runs scored by two batsmen, say, A and B, in seven completed innings.

Batsman	Score for seven completed innings						
A	20	65	0	50	100	35	80
B	40	55	35	50	65	45	60

The mean score of $A = \frac{1}{7}(20+65+0+50+100+35+80)$

$$= \frac{350}{7} = 50.$$

The mean score of $B = \frac{1}{7}(40+55+35+50+65+45+60)$,

$$= \frac{350}{7} = 50.$$

The mean score of each batsman is the same. The scores of A are much more d spersed, scattered or spread about the mean, than those of B. The scores of 100 and 80 made by A are better than any made by B. The scores, however, of zero and 20 are worse. The performances of A are more variable than those of B and hence less reliable. The purpose of the present discussion is to investigate the methods by which the dispersions of groups of observations can be compared. For this

purpose. various measures of dispersion or variation are provided, the most common being the range, the mean deviation, the standard deviation, etc. which will now be considered.

(2) Range. Definition. The range of a variate is defined as the difference between its highest and lowest given values.

In the above illustration the range of scores of A is $100 - 0 = 100$ and for B is $65 - 35 = 30$. Sometimes, the range is given by simply quoting the smallest and the largest observations such as $0 - 100$ in the case of A and $35 - 65$ in the case of B.

(3) Quartiles. Definition. *Those values which divide the set of observations, when arranged in order of magnitude, into four equal parts are called the quartiles.*

These values, denoted by Q_1, Q_2 and Q_3 are called the first, the second and the third quartiles respectively, the value Q_2 being obviously equal to the median. The first and the third quartiles are also known as the lower and upper quartiles respectively.

Interquartile range. Definition. *The interquartile range is $Q_3 - Q_1$ where Q_1 and Q_3 are the first and third quartiles.*

The Semi-interquartile range. Definition. *The semi-interquartile range (also known as the quartile deviation) is defined by* $(Q_3 - Q_1)/2$.

Method for the calculation of the lower and upper quartiles.

(i) Quartiles for ungrouped data. In this case it will be convenient in practice to have the following rules :

(a) When n, the number of observations, is not divisible by 4, Q_1 is the value of $\left(\left[\frac{n}{4}\right] + 1\right)$ th observation and Q_3 is the value of $\left(\left[\frac{3n}{4}\right] + 1\right)$ th observation, when the observations are arranged in order of magnitude and $[x]$ means the integral part contained in x;

(b) when n is divisible by 4, Q_1 is the arithmetic mean of the $\left(\frac{n}{4}\right)$th and $\left(\frac{n}{4} + 1\right)$ th observations and Q_2 is the arithmetic mean of

$\left(\frac{3n}{4}\right)$ th and $\left(\frac{3n}{4}+1\right)$ th observations, the observations being arranged in order of magnitude.

These rules are similar to the rule given earlier for the calculation of the median (i. e. Q_2) for ungrouped data.

Illustrations. Find Q_1 and Q_3 for the following two sets of observations arranged in order of magnitude.

Ex. (1). 7, 9, 11, 13, 15, 17, 20, 23, 28, 30, 35.

Here $n = 11$. $\therefore$ Q_1 = value of $\left(\left[\frac{11}{4}\right]+1\right)$ th, i. e. of the third observation. $\therefore$ $Q_1 = 11$.

Also Q_3 = value of $\left(\left[\frac{33}{4}\right]+1\right)$ th, i. e. of the ninth observation.

$\therefore$ $Q_3 = 28$.

Ex. (2). 11, 12, 14, 18, 22, 26, 30, 32.

Here $n = 8$. Hence Q_1 = the arithmetic mean of the second and third observations, since $\frac{n}{4} = 2$, and is, therefore, $= \frac{12+14}{2} = 13$.

Similarly Q_3 = the arithmetic mean of the 6th and 7th observations and is, therefore, $= \frac{26+30}{2} = 28$.

(ii) Quartiles for grouped data. In this case the formulae for Q_1 and Q_3 will be the same as the one for the calculation of the median (i.e. the middle quartile Q_2) given in article 7·2 of Chapter 13, Page, 244 provided $n/2$ is replaced by $n/4$ and $3n/4$ respectively and l_1, $(\Sigma f)_{l_1}$ and f_m refer to the corresponding intervals in which $n/4$ and $3n/4$ respectively lie. These results are, of course, obtained by linear interpolation as in the case of the median.

Illustration. Find Q_1 and Q_3 and hence obtain the interquartile range and the quartile deviation for the frequency distribution of the weekly wages, in rupees, of 65 employees given in Table 1.9 of Chapter 12, Page 229.

There are in all 65 observations. Hence the lower quartile Q_1 is that wage obtained by counting $65 \times 1/4$, i. e. 16.25 of the cases beginning from the lowest i. e. the first class. Since the first class contains 8 cases, we must take (16.25 − 8), i. e. 8.25 of the 10 cases from the second class. The lower limit l_1 of this second class is 59.995 and the size of this class is 10. Hence by using the method of linear interpolation we have the first quartile

$$Q_1 = 59.995 + \left(\frac{8.25}{10}\right) \times 10 = 68.245$$

$\therefore$ $Q_1 = 68.25$ rupees, nearly.

The third quartile Q_3 is obtained by counting the first $65 \times \frac{3}{4}$, i. e. 48.75 of the cases. Since the first four classes, comprise 48 cases, we must take $(48.75 - 48) = .75$ of the 10 cases from the fifth class having 10 as the size of the class. Here l_1 = lower limit of the fifth class and is = 89.995

Hence by linear interpolation we have the third quartile

$$Q_3 = 89.995 + \frac{.75}{10} \times 10 = 0.745;$$

i. e. $Q_3 = 90.75$ rupees, say.

Hence the interquartile range is

$Q_3 - Q_1 = 90.75 - 68.25 = 22.50$ rupees, and the quartile deviation is

$Q = \frac{1}{2}(Q_3 - Q_1) = 11.25$ rupees.

The median, i. e. the middle quartile Q_2, is obtained by counting the first $65 \times \frac{2}{4}$ i. e. 32·5 of the cases. Since the first two classes comprise 18 cases, we must take (32.5 − 18), i. e. 14.5 of the 16 cases from the third class having 16 as the frequency, 10 as the size of the class and l_1 = lower limit of this class = 69.995. Hence by linear interpolation we have the median

$$Q_2 = 69.995 + \frac{14.5}{16} \times 10 = 69.995 + 9.0625$$
$$= 79.0575;$$

$\therefore$ $Q_2 = 79.06$ rupees, say.

Note : The formulae for Q_1 and Q_3 will be similar to the formula for the median (i. e. the middle quartile Q_2) given in article 7.2 of Chapter 13, provided that $n/2$ is replaced by $n/4$ and $3n/4$ respectively and l_1, $(\Sigma f)_1$ and f_m refer to the corresponding intervals in which $n/4$ and $3n/4$ respectiveiy lie.

(4) The Mean Deviation. if a be the chosen average value of the variable x, then $x_i - a$ is the deviation of the i th given value of x from the average. Clearly the greater the deviations

$$x_1 - a,\ x_2 - a,\ \ldots\ldots,\ x_n - a$$

in magnitude, the higher is the dispersion of x. One may, therefore, consider some way of combining the deviations to get a measure of dispersion. The simple arithmetic mean of the deviations $\frac{\Sigma(x_i - a)}{n}$ cannot serve our purpose since this sum vanishes when a is the arithmetic mean. This difficulty may be overcome by considering, instead of the deviations themselves, their absolute values, in which case their magnitude alone (without the associated sign) will be taken into account.

The arithmetic mean of these absolute deviations may be taken as the required measure of dispersion.

The mean deviation. Definition. *The mean deviation (or average deviation) of a set of n numbers* $x_1, x_2, x_3, \ldots\ldots, x_n$ is defined by

$$\frac{\sum_{i=1}^{n} \left| (x_i - \bar{x}) \right|}{n}$$

where $\bar{x}$ *is the arithmetic mean of the numbers* x_i, $(i = 1, 2, \ldots\ldots, n)$ *and* $|(x_i - \bar{x})|$ *is the absolute value of the deviation of* x *from* $\bar{x}$.

We may abbreviate 'mean deviation' by 'M. D.' and write

$$\text{M.D.} = \frac{\Sigma |(x_i - \bar{x})|}{n}.$$

If $\overline{|x_i - \bar{x}|}$ denotes the arithmetic mean of $|(x_i - \bar{x})|$ for $i = 1, 2, \ldots\ldots, n$, then we can also write

$$\text{M. D.} = \overline{|(x_i - \bar{x})|}.$$

Illustration : Find the mean deviation of the numbers

2, 3, 6, 8, 11.

$$\text{Arithmetic mean } \bar{x} = \frac{2 + 3 + 6 + 8 + 11}{5} = \frac{30}{5} = 6.$$

$$\text{M. D.} = \frac{|2-6| + |3-6| + |6-6| + |8-6| + |11-6|}{5}$$

$$= \frac{4 + 3 + 0 + 2 + 5}{5} = \frac{14}{5} = 2.8.$$

Example : Find the mean deviation of the set of numbers

(i) 12, 6, 7, 3, 15, 10, 18, 5; (ii) 9, 3, 8, 8, 9, 8, 9, 18.

Ans : (i) $\bar{x} = 9.5$, M. D. = 4 25; (ii) $\bar{x} = 9$, M. D. = 2.25.

(4·1) Mean deviation of observations which are repeated.

If the observations $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots, f_n$ respectively then their mean deviation is defined as

$$\frac{\sum_{i=1}^{k} f_i |x_i - \bar{x}|}{N} \quad \text{where } N = \sum_{1}^{k} f_i.$$

This will be the same as $\overline{|x_i - \bar{x}|}$ where x_i takes all the values of the variable x for different values of i.

(4·2) Mean Deviation for Grouped Data. If $x_1, x_2, \ldots\ldots, x_k$ represent class marks and $f_1, f_2, \ldots\ldots, f_k$ the corresponding class frequencies, then the mean deviation for this grouped data is given as

$$\frac{\sum_1^k f_i \mid x_i - \bar{x} \mid}{N} \quad \text{where } N = \sum_1^k f_i.$$

Occasionally the mean deviation is defined in terms of absolute deviations from the median or other average instead of the mean. It can be shown that the mean deviation about the median is a minimum. It may be noted that it would be more appropriate to use the terminology, mean absolute deviation rather than mean deviation. However, it has become a convention simply to mention mean deviation and it always means mean absolute deviation. If the deviations are taken from any average other than the arithmetic mean, specific mention will have to be made.

Illustration : Find the mean deviation of the weights of 100 workers given in Table 1.5, of Chapter 12.

TABLE 3.1

Weight (Kg.)	Class Mark	$\mid x - \bar{x} \mid$ $\bar{x} = 67.45$	Frequency f	$f \mid x - \bar{x} \mid$
60–62	61	6.45	5	32.25
63–65	64	3.45	18	62.10
66–68	67	0.45	42	18.90
69–71	70	2.55	27	68.85
72–74	73	5.55	8	44.40
Total			100	226.50

We have already calculated $\bar{x} = 67{\cdot}45$ for this grouped data. (See article 4, Illustration 2, Table 2.5, of Chapter 13.).

$$\text{Mean deviation} = \frac{\Sigma f_i \mid (x_i - \bar{x}) \mid}{\Sigma f_i} = \frac{226.50}{100} = 2.265.$$

(5) Standard Deviation. In considering the deviations $x_i - a$ for obtaining a measure of dispersion, we may get rid of their signs by taking, instead of their absolute values, their squares $(x_i - a)^2$. Like the

absolute values, these squares will also reflect the dispersion of the variate about a. The positive square root of the arithmetic mean of the sum of these squares of deviations viz.

$$\sqrt{\frac{1}{n}\Sigma(x_i-a)^2}$$

is called the root mean square deviation about a and it may be accepted as a measure of dispersion as an alternative to the mean deviation. The measure of dispersion obtained by putting $\bar{x}$ for a is called the standard deviation of x. Thus, we have, the following definition :

Standard Deviation. Definition : *For a set of n observations* $x_1, x_2, \ldots, x_n$ *with mean* $\bar{x}$, *the standard deviation* σ (*read as sigma*) *is given by*

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}}$$

$$\text{or} \quad \sigma = \sqrt{\frac{\Sigma f_i(x_i-\bar{x})^2}{N}} \quad \text{where } N = \Sigma f_i$$

f_i *being the frequency of the variate* x_i.

Note 1 : When it is necessary to distinguish the standard deviation of a population from the standard deviation of a sample drawn from this population, we often use the symbol s for the latter and σ for the former.

Note 2 : Sometimes the standard deviation for the data of a sample is obtained by writing $(n-1)$ instead of n in the denominator of the formulae since the resulting value represents a better estimate of the standard deviation of a population from which the sample is taken. For large values of n there is practically no difference between the two expressions obtained by dividing by n or $(n-1)$. We shall, however, adhere to the definition of the standard deviation given above.

(6) The Variance. Definition. *The square of the standard deviation of a set of observations is called the variance.*

Thus, we have with the usual notation the variance

$$\sigma^2 = \frac{\sum_{1}^{n}(x_i-\bar{x})^2}{n} \quad \text{or} \quad \sigma^2 = \frac{\Sigma f_i(x_i-\bar{x})^2}{N}, \; N = \Sigma f_i$$

as the case may be.

To calculate the variance in practice, we may obtain the individual values of $(x_i - \bar{x})$, square them, add and divide by n, the total number of observations.

Illustration : The following Table shows the calculations for the variance in a simple example.

TABLE 3.2

x	$(x - \bar{x})$	$(x - \bar{x})^2$
13	2.8	7.84
8	−2.2	4.84
9	−1.2	1.44
12	1.8	3.24
9	−1.2	1.44
Total : 51	0.0	18.80

There are five observations; their total is 51.

Hence the mean $\bar{x} = \frac{51}{5}$, i. e. $\frac{102}{10}$, i. e. 10.2.

In the 2nd column we have written $x - \bar{x}$ for each observation.

In the 3rd column we have written $(x - \bar{x})^2$ for each observation.

Hence the variance, $\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{18.80}{5}$

$$= \frac{37.60}{10} = 3.76.$$

In the above illustration, the standard deviation, $\sigma_x = \sqrt{3.76}$.

(7) Short Methods for Computing the Variance and the Standard Deviation (raw data).

The procedure for actually obtaining individual values of $(x - \bar{x})$ is not very satisfactory in large samples, not only because the arithmetic becomes tedious but also because the mean of a large sample does not usually turn out to be a number which is exact to only one decimal place. If the mean has to be obtained to more than one decimal place, then all the values of $(x - \bar{x})$ must likewise be carried to two or more decimal places, and hence the values of $(x - \bar{x})^2$ must be taken to

even more decimal places, so that the arithmetical labour is still further increased. We, therefore, resort to an alternative method of calculating $\Sigma (x - \bar{x})^2$.

We have $\Sigma (x - \bar{x})^2 = \Sigma [x^2 - 2x\bar{x} + (\bar{x})^2]$

$= \Sigma x^2 - 2\bar{x}\,\Sigma x + n(\bar{x})^2$, since $\bar{x}$ is constant for a given set of observations

$= \Sigma x^2 - 2n(\bar{x})^2 + n(\bar{x})^2$ since $\Sigma x = n\bar{x}$

$= \Sigma x^2 - n(\bar{x})^2$;

$$\therefore \quad \sigma_x^2 = \frac{\Sigma (x - \bar{x})^2}{n} = \frac{\Sigma x^2}{n} - (\bar{x})^2, \text{ or } \frac{\Sigma x^2}{n} - \frac{(\Sigma x)^2}{n^2}.$$

We have, thus, the formulae

$$\sigma_x^2 = \frac{\Sigma (x_i - \bar{x})^2}{n} \quad \ldots \quad \ldots \text{ (i)}$$

$$= \frac{1}{n} [\Sigma x_i^2 - n(\bar{x})^2] \quad \ldots \quad \ldots \text{ (ii)}$$

$$= \frac{\Sigma x_i^2}{n} - (\bar{x})^2. \quad \ldots \quad \ldots \text{ (iii)}$$

Illustration : Find the standard deviation of the following observations.

8, 9, 11, 11, 12, 9, 10, 5, 8, 7.

We have, with the usual notation, the formula

$$\sigma = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n}}, \text{ where } \bar{x} = \frac{\Sigma x_i}{n} = \frac{90}{10} = 9.$$

$$\therefore \quad \Sigma (x_i - \bar{x})^2 = 1 + 0 + 4 + 4 + 9 + 0 + 1 + 16 + 1 + 4 = 40.$$

$$\therefore \quad \sigma^2 = \frac{40}{10} = 4; \qquad \therefore \quad \sigma = 2.$$

If we use formula (ii), we have

$$\sigma^2 = \frac{1}{n} [\Sigma x_i^2 - n(\bar{x})^2].$$

Now we have.

$\Sigma x_i^2 = 64 + 81 + 121 + 121 + 144 + 81 + 100 + 25 + 64 + 49 = 850$,

$\sigma^2 = \frac{1}{10} (850 - 10 \times 81)$ since $n = 10$ and $\bar{x} = 9$.

$$\therefore \quad \sigma^2 = \frac{1}{10} (40) = 4. \qquad \therefore \quad \sigma = 2.$$

If we use the formula (iii), we have

$$\sigma^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2.$$

$$\therefore \quad \sigma^2 = \frac{850}{10} - 81 = 4. \quad \therefore \quad \sigma = 2.$$

Example : Find the standard deviation of the set of numbers

12, 6, 7, 3, 15, 10, 18, 5. **Ans. 4.87.**

(8) Formulae for the variance of an Ungrouped Frequency Table. We can modify the formulae (i), (ii), (iii) of the last article to allow or frequencies corresponding to the various values of x.

We have from the definition

$$\sigma^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{N} \text{ where } N = \Sigma f_i \quad \ldots \quad \ldots \quad (i)$$

$$= \frac{\Sigma f_i [\, x_i^2 - 2x_i\bar{x} + (\bar{x})^2]}{N}$$

$$= \frac{\Sigma f_i x_i^2 - 2\bar{x}\Sigma f_i x_i + (\bar{x})^2 \Sigma f_i}{N}, \text{ since } \bar{x} \text{ is constant}$$

$$= \frac{\Sigma f_i x_i^2}{N} - 2\bar{x} \cdot \frac{\Sigma f_i x_i}{N} + (\bar{x})^2, \text{ since } \Sigma f_i = N$$

$$= \frac{\Sigma f_i x_i^2}{N} - 2(\bar{x})^2 + (\bar{x})^2, \text{ since } \bar{x} = \frac{\Sigma f_i x_i}{N}$$

$$= \frac{\Sigma f_i x_i^2}{N} - (\bar{x})^2 \quad \ldots \quad \ldots \quad (ii)$$

$$= \frac{1}{N}\left[\, \Sigma f_i x_i^2 - N(\bar{x})^2 \,\right]. \quad \ldots \quad \ldots \quad (iii)$$

Any formula which is convenient can be used. However, we have to go through tedious calculations. Short cut methods are available by changing the origin or by changing both the origin and scale simultaneously and these are discussed in article 10.

(9) The Calculations for σ^2 are not affected by a change of origin. The arithmetical labour of calculating σ^2 may be reduced by the device of subtracting a constant from each observation, i. e. by choosing a convenient origin from which all the observations are measured.

Let each observation be measured from a value, say, a. The observations then become

$$x_1 - a, x_2 - a, x_3 - a, \ldots\ldots, x_n - a.$$

Let, in general, $u = x - a$ so that $\bar{u} = \bar{x} - a$.

Now $\sigma_x^2 = \dfrac{\Sigma f_i (x_i - \bar{x})^2}{N}$

$$= \frac{\Sigma f_i [u_i + a - (\bar{u} + a)]^2}{N}$$

$$= \frac{\Sigma f_i (u_i - \bar{u})^2}{N}$$

$$= \sigma_u^2.$$

$\therefore \sigma_x = \sigma_u$.

Illustration : As an example with a large number of observations, we calculate the variance of the observations given in the Table 2.1, of Chapter 13.

TABLE 3.3

x	u $(x - 20)$	u^2	f	fu	fu^2
17	−3	9	3	−9	27
18	−2	4	6	−12	24
19	−1	1	11	−11	11
→20	0	0	20	0	0
21	1	1	8	8	8
22	2	4	1	2	4
23	3	9	1	3	9
Total			50	− 32 + 13 = − 19	83

We have $\sigma_x^2 = \sigma_u^2$

$$= \frac{\Sigma f_i u_i^2}{N} - (\bar{u})^2.$$

We have from the Table, $N = \Sigma f_i = 50$; $\bar{u} = \dfrac{\Sigma f_i u_i}{N} = \dfrac{-19}{50}$.

$$\therefore \quad \sigma_x^2 = \frac{83}{50} - \left(\frac{-19}{50}\right)^2$$

$$= \frac{166}{100} - \left(\frac{-38}{100}\right)^2$$

$$= \frac{166}{100} - \frac{1444}{10000}$$

$$= 1.66 - .1444$$

$$= 1.5156.$$

(10) Change of Origin and Scale. *If x is any variate and $u = \frac{x-a}{c}$ where a and c are constants and c is positive, then with the usual notations we have*

$$\sigma_x^2 = c^2 \sigma_u^2 \quad \text{or} \quad \sigma_x = c\,\sigma_u.$$

By definition, we have

$$\sigma_x^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{N} \text{ where } N = \Sigma f_i \text{ and } \bar{x} = \frac{\Sigma f_i x_i}{N}.$$

Here, we have $x_i = a + cu_i$, and $\bar{x} = a + c\bar{u}$.

$$\therefore \quad \sigma_x^2 = \frac{\Sigma f_i c^2 (u_i - \bar{u})^2}{N}, \quad \because \quad x_i - \bar{x} = c(u_i - \bar{u})$$

$$= c^2 \frac{\Sigma f (u_i - \bar{u})^2}{N}$$

$$= c^2 \sigma_u^2, \text{ by definition.}$$

$$\therefore \quad \sigma_x = c\,\sigma_u.$$

These are very useful formulae for calculating variance and standard deviation.

When the observations are arranged in frequency classes, the class marks are used for calculating the variance, just as they were used for the mean. However, using the class marks in this way changes the variance slightly and hence it becomes necessary to correct the variance by the use of a device known as Sheppard's adjustment. However, we do not propose to introduce Shephard's adjustment at this level.

Illustrative Examples.

Ex. 1. Find the standard deviation of the weights of the workers given in Table 1.5 of Chapter 12.

We prepare the following table.

TABLE 3.4

Class mark x_i	Frequency f_i	$u_i = \frac{x_i - a}{c}$, $a = 67,\ c = 3$	$f_i u_i$	$f_i u_i^2$
61	5	−2	−10	20
64	18	−1	−18	18
→67	42	0	0	0
70	27	1	27	27
73	8	2	16	32
Total	100		− 28 + 43 = 15	97

We have $\sigma_x^2 = c^2 \sigma_u^2$; here $c = 3$, the width of the class.

$$\therefore \quad \sigma_x^2 = 9 \left[\frac{\Sigma f_i u_i^2}{N} - (\bar{u})^2 \right]$$

$$= 9 \left[\frac{97}{100} - \left(\frac{15}{100} \right)^2 \right] \text{ since } \Sigma f_i u_i^2 = 97,$$

$$N = 100 \text{ and } \bar{u} = \frac{\Sigma f_i u_i}{N} = \frac{15}{100},$$

$$= 9 \ [.97 - (\cdot 15)^2] = 9 \ [\cdot 97 - \cdot 0225] = 9 \ [\cdot 9475].$$

$\therefore \ \sigma_x = 3 \sqrt{.9475} = 2 \cdot 92.$

Students are advised to use the square root tables for finding the roots if required.

Ex. 2. Below is given the frequency distribution according to weights of 110 parcels selected at random from a big consignment. Find the mean weight and the standard deviation.

Frequency distribution of weights of 110 parcels :

Weight (in kg)	Frequency
70—79	9
80—89	14
90—99	25
100—109	30
110—119	15
120—129	10
130—139	5
140—149	2
Total	110

We prepare the following table.

TABLE 3·5

Class mark x_i	Frequency f_i	$u_i = \frac{x_i - a}{c}$, $a = 104{\cdot}5$, $c = 10$	$f_i u_i$	$f_i u_i^2$
74·5	9	−3	−27	81
84·5	14	−2	−28	56
94·5	25	−1	−25	25
→104·5	30	0	0	0
114·5	15	1	15	15
124·5	10	2	20	40
134·5	5	3	15	45
144·5	2	4	8	32
Total	110		− 80 + 58 = − 22	294

We have, with the usual notation, the formula for the mean $\bar{x} = a + c\bar{u}$, where $a = 104{\cdot}5$, $c = 10$.

Now $\bar{u} = \frac{\Sigma f_i u_i}{N} = \frac{-22}{110} = -{\cdot}2.$

$\therefore$ $\bar{x} = 104{\cdot}5 + 10(-{\cdot}2)$

$= 104{\cdot}5 - 2{\cdot}0$

$= 102{\cdot}5.$

Now $\sigma_x^2 = c^2 \sigma_u^2$

$= c^2 \left[\frac{\Sigma f_i u_i^2}{N} - (\bar{u})^2 \right]$

$= \frac{c^2}{N} [\Sigma f_i u_i^2 - N(\bar{u})^2]$

$= \frac{100}{110} [294 - (110)(-{\cdot}2)^2]$

$= \frac{100}{110} [294 - 4.40] = \frac{100}{110} [289.60] = 100 [2.632]$

$\therefore$ $\sigma_x = 10\sqrt{2{\cdot}632} = 10 \times 1.622 = 16.22.$

Calculations may be done as per convenience correct to 2 or 3 places of decimals.

(11) Comparison of Range, Mean Deviation and Standard Deviation. The range becomes meaningless when at least one of the two limits of the variable is infinite. The range, however, is the easiest to compute. The other two require almost the same amount of computational labour. Both the mean deviation and the standard deviation are based on all the given values of the variable and hence they characterise the whole set of values; the range is inferior in this respect. The range often fails to give a proper idea of the dispersion depending as it does only on the two extreme values of the variable. The standard deviation may be generally regarded as the best measure of dispersion, just as the arithmetic mean generally serves as the best measure of central tendency.

EXERCISE 14

(1) The following data gives the percentage of the S. S. C. examination marks of 20 students :

65, 72, 67, 69, 70, 66, 71, 73, 74, 75,
71, 65, 67, 73, 69, 74, 66, 75, 68, 70.

Find the mean and the standard deviation of the percentage of marks.

(2) The annual salaries of a group of employees are given in the following table :

Salaries in hundreds of rupees	45	50	55	60	65	70	75	80
No. of persons	3	5	8	7	9	7	4	7

Calculate the mean and the standard deviation of the salaries.

(3) Find the mean and standard deviation of the following distribution :

Class	3 to 4·9	5 to 6·9	7 to 8·9	9 to 10·9	11 to 12·9	13 to 14·9	15 to 16·9	Total
Frequency	6	7	32	81	44	25	5	200

(4) A frequency distribution with class intervals of equal lengths has resulted into the following distribution on changing the origin and scale :

u	−4	−3	−2	−1	0	1	2	3	Total
f	2	5	8	18	22	13	8	4	80

Find the new origin and the length of the class interval if the arithmetic mean and the standard deviation of the original distribution were known to be 135·3 and 9·6 respectively.

(5) Find the mean and standard deviation from the following distribution of income of 70 workers in a factory :

Income in rupees	Number of workers
60 – 70	6
70 – 80	7
80 – 90	10
90 – 100	20
100 – 110	12
110 – 120	10
120 – 130	5
Total	70

(6) The following is the frequency table showing heights of 200 boys :

Height in inches	53	55	57	59	61	63	65	67
No. of students	4	17	28	63	46	22	7	13

Find the mean and standard deviation.

(7) Below is given the frequency distribution of monthly earnings of 80 sweepers employed in a certain municipality selected at random at a district place. Find the mean and standard deviation for the frequency distribution.

Earning in rupees	Persons	Earning in rupees	Persons
60 – 64	2	85 – 89	10
65 – 69	4	90 – 94	8
70 – 74	4	95 – 99	5
75 – 79	15	100 – 104	5
80 – 84	25	105 – 109	2

(8) The mean and the standard deviation of a group of n observations are $\bar{x}$ and σ_1 respectively. A new observation X is added to the group. Find the standard deviation of the $(n + 1)$ observations.

(9) The mean and the standard deviation of a group of 25 observations were calculated to be 20 and 5 respectively. It was later discovered that there was an error in recording one observation. It was recorded as 8. Find the mean and standard deviation of the group after omitting this observation.

(10) Calculate the standard deviation for the following data :

Intelligence quotient class Interval	Frequency	Intelligence quotient class interval	Frequency
60— 70	2	110—120	36
70— 80	7	120—130	18
80— 90	12	130—140	10
90—100	28	140—150	4
100—110	42	150—160	1

(11) Find the range, the first and the third quartiles and the interquartile range for the data in the following table :

x	462	480	498	516	534	552	570	588	606	624
f	98	75	56	42	30	21	15	11	6	2

(12) Find the three quartiles, the interquartile range and the quartile deviation for the distribution of the weights of 100 factory workers given in table 1·5 of chapter 12.

(13) The following table shows the distribution of maximum loads in short tons supported by certain cables. Find the interquartile range.

Load (Short tons)	9·3 to 9·7	9·8 to 10·2	10·3 to 10·7	10·8 to 11·2	11·3 to 11·7	11·8 to 12·2	12·3 to 12·7	12·8 to 13·2	Total
No. of Cables	2	5	12	17	14	6	3	1	60

(14) The following table shows a frequency distribution of marks in Mathematics obtained by 120 students. Find the three quartiles of the distribution and interpret the significance of each.

Marks	30-39	40-49	50-59	60-69	70-79	80-89	90-100
No. of Students	1	3	11	21	43	32	9

(15) Fill in the blanks in the following :—

(i) The attendance of a class of 47 students for 10 days is
40, 42, 39, 35, 45, 41, 45, 44, 38 and 41.
$\therefore$ the mean deviation of the attendance is.....

(ii) The variance of the set of observations
3, 4, 5, 7, 9, 10, 11 is.....

(iii) The standard deviation for the set of observations
1, 2, 3, 4, 5, 6, 7 is.....

(iv) If $\Sigma f_i = 100$, $\Sigma f_i x_i = 200$ and $\Sigma f_i x_i^2 = 800$,
then the mean and standard deviation, respectively, are..... , ...

(16) State the correct answer from the given alternatives in the following :—

(i) The standard deviation of a set of 7 observations is 10. If 3 is added to each observation, then the standard deviation of the new observations is (a) 3, (b) 7, (c) 10, (d) 13.

(ii) The standard deviation of the observations 3, 5, 7, 8, 10, 11 is σ.
The standard deviation of the observations obtained by adding 2 to each observation is
(a) the same as σ, (b) not equal to σ.

(iii) Two groups of 10 different observations have the same standard deviation but have different arithmetic means namely 100 and 80. Hence the variance of the first group of observations, is (a) smaller than that of the second, (b) greater than that of the second, (c) equal to that of the second.

ANSWERS

(1) Mean = 70, S. D. = 3·25. (2) Mean = Rs. 6,360, S. D. = Rs. 1,035.

(3) Mean = 10·4, S. D. = 2.42. (4) New origin : 136·5, Length of the class : 6.

(5) Mean = Rs. 95·714, S. D. = Rs. 16·419.

(6) Mean = 59.89'', S. D. = 3.154''.

(7) Mean = Rs. 83.75, S. D. = Rs. 9.78.

(8) $\sigma^2 = \frac{n\sigma_1^2}{n+1} + \frac{n}{(n+1)^2}(\bar{x} - X)^2$.

(9) Mean = 20.5, S. D. = 4.448.

(10) S. D. = $10\sqrt{109}$.

(11) Range = 462 – 624, $Q_1 = 462$, $Q_3 = 516$, interquartile range = 54.

(12) $Q_1 = 65.64$ kg., $Q_2 = 67.43$ kg., $Q_3 = 69.61$ kg.,
interquartile range = 3.97 kg. quartile deviation 1.985 kg.

(13) 99 short tons.

(14) $Q_1 = 67$, $Q_2 = 75$, $Q_3 = 83$; 25% scored 67 or lower, 50% scored 75 or lower and 75% scored 83 or lower.

(15) (i) 2.4, (ii) $8\frac{2}{7}$, (iii) 2. (iv) 2, 2.

(16) (i) (c), (ii) (a), (iii) (c).

□ □ □

Summary of Important Results

Chapter 1. Measurement of Angles.

(1) A directed angle is an ordered pair (initial ray, terminal ray) of co-terminal rays associated with anti-clockwise or clockwise rotation from the initial ray giving the + ve or the − ve measure respectively of the directed angle.

(2) In the sexagesimal system, we have
1 rt. angle = 90°, $1^\circ = 60'$ and $1' = 60''$.

(3) In the circular measure (system) a radian is defined as the measure of the angle subtended at the centre of a circle by its arc equal to its radius.

(4) Conversion Formulae

(i) $\pi^c = 180^\circ$, i.e. $1^c = \left(\frac{180}{\pi}\right)^\circ \doteqdot 57^\circ\, 17'\, 45''$.

(ii) $1^\circ = \left(\frac{\pi}{180}\right)^c$.

(5) $s = r\theta$, where s is the arc length, r the radius of the circle and θ the radian measure of the angle subtended at the centre of the circle by its arc of length s.

(6) $\frac{1}{2} r^2 \theta$ is the area of a sector, of a circle of radius r, bounded by radii including an angle θ in radian measure.

Chapter 2. Trigonometrical Functions.

(1) If $P(x, y)$ is any point in OA, referred to the rectangular axes OX, OY and $m \angle XOA = \alpha$ and $OP = r$, then
$\cos\alpha = x/r$; $\sin\alpha = y/r$; $\tan\alpha = y/x$; $x \neq 0$;
$\sec\alpha = r/x$, $x \neq 0$; $\operatorname{cosec}\alpha = r/y$, $y \neq 0$; $\cot\alpha = x/y$, $y \neq 0$.

(2) We have by definition

$$\sec\alpha = \frac{1}{\cos\alpha},\ \cos\alpha \neq 0;\ \operatorname{cosec}\alpha = \frac{1}{\sin\alpha},\ \sin\alpha \neq 0;\ \cot\alpha = \frac{1}{\tan\alpha},\ \tan\alpha \neq 0.$$

(3) If $0 < \alpha < \frac{\pi}{2}$, then all the trigonometric ratios are + ve.

If $\frac{\pi}{2} < \alpha < \pi$, then $\sin\alpha$ and $\operatorname{cosec}\alpha$ alone are + ve.

If $\pi < \alpha < \frac{3\pi}{2}$, then $\tan\alpha$ and $\cot\alpha$ alone are + ve.

If $\frac{3\pi}{2} < \alpha < 2\pi$, then $\cos\alpha$ and $\sec\alpha$ alone are + ve.

(4) If $P(x, y)$ is a point on the unit circle with its centre at O so that $OP\ (= r) = 1$, then we have

$$\cos\alpha = x,\quad \sin\alpha = y.$$

(i) If $\alpha = 0$, then $x = 1$, $y = 0$, $\therefore$ $\cos 0 = 1$, $\sin 0 = 0$.

(ii) If $\alpha = \pi/2$, then $x = 0$, $y = 1$, $\therefore$ $\cos \pi/2 = 0$, $\sin \pi/2 = 1$.

(iii) If $\alpha = \pi$, then $x = -1$, $y = 0$, $\therefore$ $\cos \pi = -1$, $\sin \pi = 0$.

(iv) If $\alpha = 3\pi/2$, then $x = 0$, $y = -1$, $\therefore$ $\cos 3\pi/2 = 0$, $\sin 3\pi/2 = -1$.

(v) If $\alpha = \pi/6$, then $x = \sqrt{3}/2$, $y = 1/2$ $\therefore$ $\cos \pi/6 = \sqrt{3}/2$, $\sin \pi/6 = 1/2$.

(vi) If $\alpha = \pi/4$, then $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$, $\therefore$ $\cos \pi/4 = 1/\sqrt{2}$, $\sin \pi/4 = 1/\sqrt{2}$.

(vii) If $\alpha = \pi/3$, then $x = 1/2$, $y = \sqrt{3}/2$, $\therefore$ $\cos \pi/3 = 1/2$, $\sin \pi/3 = \sqrt{3}/2$.

(viii) If $\alpha = 3\pi/4$, then $x = -1/\sqrt{2}$, $y = 1/\sqrt{2}$, $\therefore$ $\cos 3\pi/4 = -1/\sqrt{2}$, $\sin 3\pi/4 = 1/\sqrt{2}$.

(ix) If $\alpha = 2\pi/3$, then $x = -1/2$, $y = \sqrt{3}/2$, $\therefore$ $\cos 2\pi/3 = -1/2$, $\sin 2\pi/3 = \sqrt{3}/2$.

The other ratios, when they are defined, can be obtained from the sine and cosine ratios.

(5) $\sin^2 \alpha + \cos^2 \alpha = 1$, $1 + \tan^2 \alpha = \sec^2 \alpha$, $1 + \cot^2 \alpha = \operatorname{cosec}^2 \alpha$.

(6) $|\sin \alpha| \leqslant 1$, $|\cos \alpha| \leqslant 1$, $|\operatorname{cosec} \alpha| \geqslant 1$, $|\sec| \geqslant 1$, $-\infty < \tan \alpha, \cot \alpha < \infty$.

(7) $\sin \alpha$, $\cos \alpha$, $\operatorname{cosec} \alpha$, $\sec \alpha$ are periodic with the period 2π, whereas $\tan \alpha$ and $\cot \alpha$ are periodic with the period π.

Chapter 3. Distance and Section Formulae.

(1) If (r, θ) are the polar co-ordinates of the point $P(x, y)$ then (a) $x = r \cos \theta$, $y = r \sin \theta$

(b) $r = \sqrt{x^2 + y^2}$ and θ is determined by $\cos \theta = x/r$ and $\sin \theta = y/r$.

(2) $P(x, y)$ is any point, then $OP = \sqrt{x^2 + y^2}$.

(3) $P(x_1, y_1)$, $Q(x_2, y_2)$ are any two points, then $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

(4) If $P(x, y)$ divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m_1 : m_2$ then

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}.$$

The mid-point of the join of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The centroid of the triangle formed by (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Chapter 4. Locus and Translation of axes.

(1) Locus is a set of points possessing a given property.

(2) Equation of the locus of a point is the equation satisfied by the co-ordinates (x, y) of every point of the locus and by the co-ordinates of no other point.

(3) If (x, y) are the co-ordinates of the point P referred to the frame (Ox, Oy) and (X, Y) are the co-ordinates of the same point P referred to the parallel frame $(O'X)$ $(O'Y)$ where $O' \equiv (h, k)$ referred to the frame (Ox, Oy) then

$$x = h + X,\ y = k + Y;\ X = x - h,\ Y = y - k.$$

Chapter 5. Straight Lines.

(1) If (x_1, y_1) and (x_2, y_2) are two points in a line with inclination θ, then its slope is

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1},\ x_1 \neq x_2.$$

If $x_1 = x_2$ then $\theta = \pi/2$ and the slope is not defined.

If $y_1 = y_2$ then $\theta = 0$ and the slope is zero.

(2) Equation of a line

(i) along the x axis : $y = 0$; parallel to the x-axis : $y = k$,

(ii) along the y-axis : $x = 0$; parallel to the y-axis : $x = h$,

(iii) in the slope origin form : $y = mx$,

(iv) in the slope intercept form : $y = mx + c$,

(v) in the slope point form : $y - y_1 = m(x - x_1)$,

(vi) in the two points form : $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$,

(vii) in the double intercepts form : $\frac{x}{a} + \frac{y}{b} = 1$,

(viii) in the normal form : $x\cos\alpha + y\sin\alpha = p$,

(ix) in the parametric form : $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$, r being the parameter.

(3) The linear equation in x and y, namely $Ax + By + C = 0$, represents a straight line and every straight line is represented by a first degree equation of the form $Ax + By + C = 0$.

(4) The line $Ax + By + C = 0$ has

(i) slope $m = -A/B$, $= -\frac{\text{coefficient of } x}{\text{coefficient of } y}$, $B \neq 0$,

(ii) x – intercept $= -C/A$, $A \neq 0$; y – intercept $= -C/B$, $B \neq 0$.

(5) The equation of a line not passing through the origin is of the form $lx + my = 1$ and thus contains only two independent constants at the most.

(6) The length of the perpendicular from the origin to the line $Ax + By + C = 0$ where $C > 0$, is $\frac{C}{\sqrt{A^2 + B^2}}$ and its inclination α is given by $\cos\alpha = \frac{A}{-\sqrt{A^2 + B^2}}$ and $\sin\alpha = \frac{B}{-\sqrt{A^2 + B^2}}$.

(7) The join of $A(x_1, y_1)$ and $B(x_2, y_2)$ is divided by the line $ax + by + c = 0$ in the ratio $= -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$; this ratio is – ve or

+ve according as the points lie on the same side or on opposite sides of the line. The expression $ax + by + c$ changes its sign as the point (x, y) crosses the line $ax + by + c = 0$ from one side to the other.

(8) The length of the perpendicular from (x_1, y_1) to the line $ax + by + c = 0$ is $\pm \dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$.

(9) The perpendicular distance between two parallel lines $ax + by + c = 0$ and $ax + by + c' = 0$ is $\dfrac{|c - c'|}{\sqrt{a^2 + b^2}}$.

(10) If θ is the acute angle between two lines with slopes m and m', then $\tan\theta = \left|\dfrac{m - m'}{1 + mm'}\right|$. The lines are parallel if $m = m'$, and perpendicular if $mm' = -1$.

(11) If θ is the acute angle between two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$, then $\tan\theta = \left|\dfrac{a'b - ab'}{aa' + bb'}\right|$.

The lines are identical if $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.

The lines are parallel if $a'b - ab' = 0$, i.e. if $\dfrac{a}{a'} = \dfrac{b}{b'} \neq \dfrac{c}{c'}$.

The lines are mutually perpendicular if $aa' + bb' = 0$.

(12) The point of intersection of the two non-parallel lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ is $\left(\dfrac{bc' - b'c}{ab' - a'b}, \dfrac{ca' - c'a}{ab' - a'b}\right)$.

(13) The three lines $a_rx + b_ry + c_r = 0$, $r = 1, 2, 3$ are concurrent if $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$.

(14) The equation of a line parallel and perpendicular to a given line $ax + by + c = 0$, can be respectively taken as $ax + by + c_1 = 0$ and $bx - ay + c_2 = 0$ where c_1 and c_2 can be determined from the given additional conditions.

Chapter 6. Formulae for Compound Angles.

(a) **Addition and Subtraction Formulae.**

(1) $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.

(2) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$.

(3) $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$.

(b) **Allied Angle Formulae.**

(1) $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$.

(2) $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$.

(3) $\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta,\ \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$

(4) $\sin(\pi-\theta)=\sin\theta,\ \cos(\pi-\theta)=-\cos\theta.$

(5) $\sin(\pi+\theta)=-\sin\theta,\ \cos(\pi+\theta)=-\cos\theta.$

(6) $\sin(\theta\pm 2n\pi)=\sin\theta,\ \cos(\theta\pm 2n\pi)=\cos\theta.$

(7) $\tan(\theta\pm n\pi)=\tan\theta.$

(c) Sum and Difference Formulae.

(1) $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}.$

(2) $\sin C-\sin D=2\sin\frac{C-D}{2}\cos\frac{C+D}{2}.$

(3) $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}.$

(4) $\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$

$=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}.$

(d) Product Formulae.

(1) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(2) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(3) $2\cos A\cos B=\cos(A-B)+\cos(A+B)$

(4) $2\sin A\sin B=\cos(A-B)-\cos(A+B).$

(e) Double Angle Formulae.

(1) $\sin 2A=2\sin A\cos A=\frac{2t}{1+t^2}$, where $t=\tan A.$

(2) $\cos 2A=\cos^2A-\sin^2A=2\cos^2A-1=1-2\sin^2A$

$=\frac{1-t^2}{1+t^2}$, where $t=\tan A.$

(3) $\tan 2A=\frac{2t}{1+t^2}$, $t=\tan A.$

(f) Triple Angle Formulae.

(1) $\sin 3A=3\sin A-4\sin^3A.$

(2) $\cos 3A=4\cos^3A-3\cos A.$

(3) $\tan 3A=\frac{3t-t^3}{1-3t^2}$, $t=\tan A.$

(g) Half Angle Formulae.

(1) $\sin\frac{A}{2}=\pm\sqrt{\frac{1}{2}(1-\cos A)}.$

(2) $\cos \frac{A}{2} = \pm \sqrt{\frac{1}{2}(1+\cos A)}$.

(3) $\tan \frac{A}{2} = \pm \sqrt{\frac{1-\cos A}{1+\cos A}}$.

The ambiguity of sign in each case can be determined if we know the quadrant in which the angle $A/2$ lies.

Chapter 7. Properties of a triangle.

(1) The Sine Rule : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

(2) The Cosine Rule : $a^2 = b^2 + c^2 - 2bc \cos A$.

(3) The Projection Rule : $a = b \cos C + c \cos B$.

(4) The Tangent Law : $\tan \left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \tan \left(\frac{A+B}{2}\right)$.

(5) (i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, where $2s = a + b + c$.

(ii) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $2s = a + b + c$.

(iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $2s = a + b + c$.

(6) Area of a $\triangle ABC$, say, $\triangle = \frac{1}{2} bc \sin A$

$= \sqrt{s(s-a)(s-b)(s-c)}$.

(7) Circum-radius $R = \frac{abc}{4\triangle}$.

(8) In-radius $r = \frac{\triangle}{s}$, $2s = a + b + c$.

(9) The angle of elevation is the angle between the horizontal ray and the ray of line of sight when the object is at a higher level.

(10) The angle of depression is the angle between the horizontal ray and the ray of line of sight when the object is at a lower level.

Chapter 8. Vectors.

(1) A directed line segment from A to B, denoted by AB, where A and B are respectively the initial and terminal points is called a vector.

(2) Scalar quantities or scalars are quantities which have only magnitude but no direction.

(3) Vectors are physical quantities which have magnitude, direction and which obey rules of vector Algebra.

(4) The definitions of the following terms should be noted. (i) magnitude of a vector, (ii) a null vector, (iii) a unit vector, (iv) negative of a vector, (v) like and unlike vectors, (vi) collinear and coplanar vectors (vii) free and localized vectors, (iii) equal vectors.

(5) The triangle law of addition. In any $\triangle ABC$, $\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$.

(6) The parallelogram law of addition : In any $\square ABCD$, $\mathbf{AB} + \mathbf{AD} = \mathbf{AC}$.

(7) Commutative property : $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(8) Associative Property : $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(9) Directed line segment OP ($= \mathbf{p}$) is the position vector of the point P w. r. t. the origin O.

(10) PQ = position vector of Q — position vector of $P = \mathbf{q} - \mathbf{p}$.

(11) Section Formula. The position vector $\mathbf{p}$ of the point P dividing seg AB in the ratio $m : n$ is given by

$\mathbf{p} = \dfrac{m\mathbf{b} + n\mathbf{a}}{m + n}$, where a and b are the position vectors of A and B.

(12) The necessary and sufficient condition for three distinct points A, B, C with position vectors a, b, c respectively, to be collinear is that there should exist three non zero scalars l, m, n, s. t. $l\mathbf{a} + m\mathbf{b} + n\mathbf{c} = 0$ where $l + m + n = 0$.

(13) If $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$, then $x\mathbf{a}$ and $y\mathbf{b}$ are called the components of r in the directions of a and b respectively and the vector r is said be resolved or decomposed in the directions of a and b.

(14) If $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$, then r can be represented by the ordered pair

$$(x, y) \text{ or } \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } [x, y].$$

(15) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ where i and j are unit vectors along the rectangular co-ordinate axes Ox, Oy then $|\mathbf{r}| = \sqrt{x^2 + y^2}$.

(16) The magnitude of the resultant of two vectors a and b including an angle α is $\sqrt{a^2 + b^2 + 2ab\cos\alpha}$ and the direction of this resultant makes an angle θ with the direction of a s. t.

$$\tan\theta = \frac{b\sin\alpha}{a + b\cos\alpha}.$$

Chapter 9. Prisms and Cylinders.

(1) The length d of a diagonal of a rectangular $\overline{\text{IP}}$ of co-terminal edges x, y, z is given by $d^2 = x^2 + y^2 + z^2$.

(2) The volume of a rectangular $\overline{\text{IP}}$ with coterminus edges x, y, z, is xyz and its surface area is $2(xy + yz + zx)$.

(3) The lateral surface area of a prism = (length of a lateral edge) × (perimeter of a right section.)

(4) The lateral surface area of a right prism = (altitude) × (perime the base).

(5) Total surface area of a prism = (lateral area) + 2 (area of a base

(6) Volume of a prism = (area of a base) × (altitude).

(7) The lateral surface area of a cylinder = (perimeter of a right section) × (generator).

(8) The lateral surface area of a right circular cylinder of base radius r and height h is $2\pi rh$ and its total surface area is $2\pi rh + 2\pi r^2 = 2\pi r(r+h)$.

(9) Volume of a cylinder = (area of base) × (altitude).

(10) Volume of a right circular cylinder of base radius r and altitude h is $\pi r^2 h$.

Chapter 10. Pyramids and Cones.

(1) The areas of parallel sections of a pyramid (or a cone) are to each other as the squares of their distances from the vertex.

(2) The lateral area of a regular pyramid = $\frac{1}{2}$ (slant height) × (perimeter of the base polygon)

(3) The volume of a pyramid = $\frac{1}{3}$ (area of base) × (altitude).

(4) The lateral curved surface of a right circular cone = πrl, where r is the base radius and l the length of a generator.

(5) The total surface of a right circular cone = $\pi r^2 + \pi rl$.

(6) The volume of a right circular cone = $\frac{1}{3}\pi r^2 h$.

Chapter 11. Sphere.

(1) The volume of a sphere = $\frac{4}{3}\pi r^3$.

(2) The surface area of a sphere = $4\pi r^2$.

(3) The volume of a spherical segment of two bases (frustrum) of bases of radii r_1 and r_2 and of height h is $\frac{\pi h}{6}(3r_2^2 + 3r_1^2 + h^2)$.

(4) The volume of a spherical cap of height h, of a sphere of radius r, is $\frac{\pi h^2}{3}(3r - h)$ and is equal to $\frac{\pi h}{6}(3a^2 + h^2)$, where a is the radius of the base of the cap.

(5) If r is the radius of a sphere of volume V and surface area S, then

$$V = \frac{1}{3} rS.$$

Chapter 12. Frequency Distributions.

(1) Population, in statistics, is any collection of objects taken for a statistical enquiry regarding a certain property of its members.

(2) A variate is a quantitative character of individuals in a population which can be expressed numerically.

(3) An attribute is a qualitative character of individuals in a population.

(4) Size of class or class width is the difference between the lower limits (or the upper limits) of any two consecutive classes.

(5) Class mark (or mid-value)
$= \frac{1}{2}$ (lower limit + upper limit)
$= \frac{1}{2}$ (lower boundary + upper boundary).

(6) Cumulative frequencies are of two types (i) less than type, (ii) greater than type.

(7) Frequency distribution can be represented graphically by (i) a frequency bar chart, (ii) a frequency polygon, (iii) a histogram, (iv) a frequency curve, (v) an ogive.

Chapter 13. Measures of Central Tendency.

(1) A single value which can be considered as typical or representative of a set of observations and around which the observations can be considered as clustered is called an average.

(2) (i) $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ gives the arithmetic mean of n observations x_i, $i = 1, 2, \ldots, n$.

(ii) $\bar{x} = \frac{1}{\sum_{1}^{n} f_i} \sum_{i}^{n} f_i x_i$ gives the arithemetic mean of observations x_i, with frequencies f_i, $i = 1, 2, \ldots, n$.

(3) If $x_1, x_2, \ldots, x_n$ are n values of the class marks with frequencies $f_1, f_2, \ldots, f_n$ respectively and if each x_i is expressed in terms of a new variable u_i by the relation $x_i = a + cu_i$, then,

$\bar{x} = a + c\,\bar{u}$ where c is the common width of all the class intervals.

(4) Median of a set of observations is the middle value (or the arithemetic mean of the two middle values when the observations are even in number) when they are arranged in order of magnitude.

(5) The median for a grouped data is

$$l_1 + \left[\frac{\frac{n}{2} - (\Sigma f)_{l_1}}{f_m} \right] c.$$

(6) The mode of a set of observations is the value of the observation having maximum frequency.

Chapter 14. Measures of Dispersion.

(1) The degree to which numerical data tends to spread about an average value is called the variation or dispersion of the data.

(2) The range of a variate is the difference between its highest and lowest given values.

(3) Those values which divide the set of observations, when arranged in order of magnitude, into four equal parts are called the quartiles and are denoted by Q_1, Q_2 and Q_3 where Q_2 is obviously equal to the median.

(4) Inter-quartile range of the distribution is $Q_3 - Q_1$ where Q_1 and Q_3 are the first and third quartiles. See article 3 page 254 for calculations.

(5) Mean deviation about the mean is

(i) $\frac{1}{n} \sum_{i=1}^{n} \left| x_i - \bar{x} \right|$, for ungrouped data

(ii) $\frac{1}{N} \sum_{i=1}^{n} f_i \left| x_i - \bar{x} \right|$, for grouped data, $\left(N = \sum_{1}^{n} f_i \right)$.

(6) Variance σ^2 of a set of observations is

(i) $\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - (\bar{x})^2$, for ungrouped data,

(ii) $\frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = \frac{1}{N} \Sigma f_i x_i^2 - (\bar{x})^2$, for grouped data,

$N = \sum_{i=1}^{n} f_i$.

(7) If x is any variate and $u = \frac{x - a}{c}$, where a and c are constants and $c > 0$, then $\sigma^2_x = c^2 \sigma^2_u$ or $\sigma_x = c \sigma_u$.

□ □ □

Use of Trigonometric Tables

Students are already familiar with the use of Logarithmic Tables of positive numbers to the base 10. We now explain, here, how to refer to the tables of natural sines and tangents. It is sufficient to have tables of sines and tangents for acute angles since the remaining ratios of other angles can be expressed in terms of sine and tangent ratios of acute angles.

In the first column of sines table, we have the measure of angles at an interval of 1°. The next column headed by zero gives the sines of these angles in the same row. The following nine columns give the sines at an interval of 6′. The remaining five columns headed by Mean differences give the differences at intervals of 1′.

Illustration. Find (i) sin 35° 42′, (ii) sin 35° 46′, (iii) sin 35° 42′ 36″.

(i) Sin 35° 42′ can be found in the row of 35° and the column headed by 42′ and is ·5835.

(ii) Sin 35° 46′ can be found by taking the number .5835 in the row of 35° and column headed by 42′ and adding to it a number ·0010 found in the same row and column headed by 4′ in the mean differences.

Thus sin 35° 46′ = ·5835 + .0010 = ·5845.

(iii) Sin 35° 42′ 36″ can be obtained by interpolation as follows.

We have sin 35° 42′ = ·5835 and sin 35° 43′ = .5837 Observe that the mean difference increase is ·0002 for an inccease of 60″.

∴ the mean difference increase for an increase of 36″ by simple proportion will be $\frac{36}{60} \times .0002 = .0001$ correct to four decimal places.

∴ sin 35° 42′ 36″ = .5835 + .0001 = ·5836.

Remark 1. The tangent tables are to be used exactly as the sine tables.

Thus, tan 25° 36′ = .4791, tan 25° 38′ = .4791 + .0007 = .4798,

$$\tan 25° 36' 30'' = .4791 + \frac{30}{60} (.0004) = .4793.$$

Remark 2. The values of the cosines and cotangents of acute angles can be obtained by using the formula $\cos A = \sin (90° - A)$ and $\cot A = \tan (90° - A)$ and using the tables of sines and tangents.

Thus cos 55° 18′ = sin 34° 42′ = .5693.
cot 37° 24′ = tan 52° 36′ = 1·3079.

Deg	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
°															
0	.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	.0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	.0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	.0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
°															
5	.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	.1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	.1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	.1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	.1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
°															
10	.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	.1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	.2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	.2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	.2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
°															
15	.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	.2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	.2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	.3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	.3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
°															
20	.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	.3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	.3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	.3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	.4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
°															
25	.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	.4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	.4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	.4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	.4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
°															
30	.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	.5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	.5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	.5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	.5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
°															
35	.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	.5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	.6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	.6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	.6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
°															
40	.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	.6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	.6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	.6820	6833	6845	6858	6871	6884	6898	6900	6921	6934	2	4	6	8	11
44	.6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

Natural cosines can be found using relation, $\cos\theta = \sin(90 - \theta)$

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′ 2′ 3′	4′ 5′
45°	.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2 4 6	8 10
46	.7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2 4 6	8 10
47	.7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2 4 6	8 10
48	.7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2 4 6	8 10
49	.7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2 4 6	8 9
50°	.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2 4 6	7 9
51	.7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2 4 5	7 9
52	.7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2 4 5	7 9
53	.7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2 3 5	7 9
54	.8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2 3 5	7 8
55°	.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2 3 5	7 8
56	.8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2 3 5	6 8
57	.8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2 3 5	6 8
58	.8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2 3 5	6 8
59	.8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1 3 4	6 7
60°	.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1 3 4	6 7
61	.8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1 3 4	6 7
62	.8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1 3 4	5 7
63	.8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1 3 4	5 6
64	.8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1 3 4	5 6
65°	.9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1 2 4	5 6
66	.9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1 2 3	5 6
67	.9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1 2 3	4 6
68	.9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1 2 3	4 5
69	.9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1 2 3	4 5
70°	.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1 2 3	4 5
71	.9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1 2 3	4 5
72	.9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1 2 3	3 4
73	.9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1 2 2	3 4
74	.9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1 2 2	3 4
75°	.9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1 1 2	3 4
76	.9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1 1 2	3 3
77	.9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1 1 2	3 3
78	.9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1 1 2	2 3
79	.9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1 1 2	2 3
80°	.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0 1 1	2 2
81	.9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0 1 1	2 2
82	.9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0 1 1	2 2
83	.9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0 1 1	1 2
84	.9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0 1 1	1 2
85°	.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0 0 1	1 1
86	.9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0 0 1	1 1
87	.9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0 0 0	1 1
88	.9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0 0 0	0 0
89	.9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0 0 0	0 0
90	1.000											

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
0°	.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
'	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	.0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5°	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	.1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	.1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10°	.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	.1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	.2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	.2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	.2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15°	.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	.2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	.3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	.3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	.3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20°	.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	.3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	.4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	.4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	.4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25°	.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	.4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	.5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	.5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	.5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30°	.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	.6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	.6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	.6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	.6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35°	.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	.7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	.7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	.7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	.8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40°	.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	.8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	.9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	.9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

Natural cotangents can be found using relation cot θ = tan (90 — θ)

Degrees.	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1′	2′	3′	4′	5′
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1.1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1.1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50°	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1.2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1.2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1.3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1.3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55°	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1.4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1.5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1.6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60°	1.7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1.9626	9711	9797	9883	9970	$\bar{0}057$	$\bar{0}145$	$\bar{0}233$	$\bar{0}323$	$\bar{0}413$	15	29	44	58	73
64	2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65°	2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2.2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2.3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2.4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2.6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70°	2.7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2.9042	9208	9375	9544	9714	9887	$\bar{0}061$	$\bar{0}237$	$\bar{0}415$	$\bar{0}595$	29	58	87	116	145
72	3.0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3.2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3.4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75°	3.7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4.0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4.3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4.7046	7453	7867	8288	8716	9152	9594	$\bar{0}045$	$\bar{0}504$	$\bar{0}970$					
79	5.1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80°	5.6713	7297	7894	8502	9124	9758	$\bar{0}405$	$\bar{1}066$	$\bar{1}742$	$\bar{2}432$					
81	6.3138	3859	4596	5350	6122	6912	7720	8548	9395	$\bar{0}264$					
82	7.1154	2066	3002	3962	4947	5958	6996	8062	9158	$\bar{0}285$					
83	8.1443	2636	3863	5126	6427	7769	9152	$\bar{0}579$	$\bar{2}052$	$\bar{3}572$					
84	9.5144	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20					
85°	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					
90	∞														

Mean differences cease to be sufficiently accurate.

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